Mathematics (MATH113)

Week 1 (Lectures No. 1 & 2)

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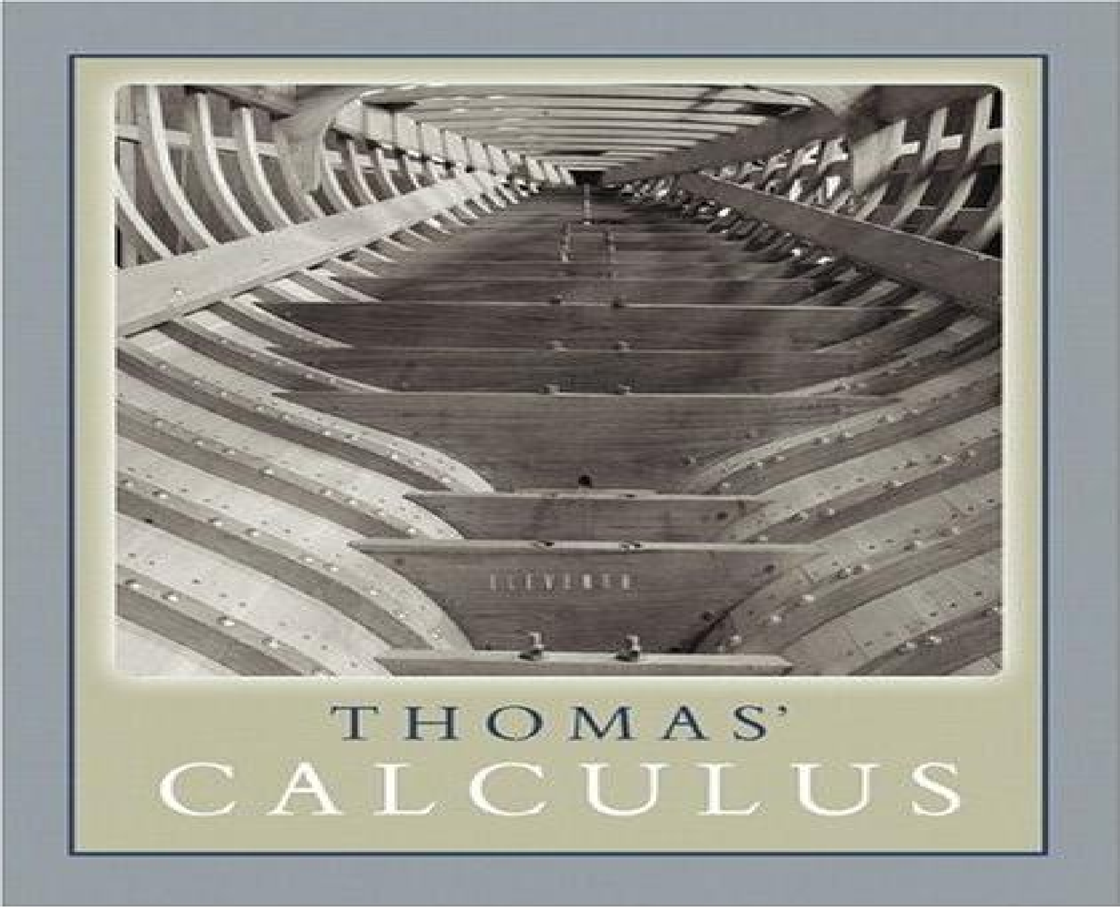


G. B. Thomas, M. D. Weir, J. Hass, and F. R. Giordano, Thomas' Calculus, 11 ed., Addison-.Wesley, 2005

What is Calculus?

"advanced algebra and geometry":
setting up Mathematics as a formal language

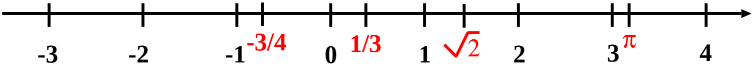
- fundamental: real numbers
- study of functions of real variables
- geometric view: graph of a function
  - continuity properties
  - slope $\leftrightarrow$ derivative
  - area $\leftrightarrow$ integral
- many techniques, based on algebraic manipulations
- many applications in all branches of modern society

Real numbers and the real line

think of the real numbers, e.g., as all decimals

examples: $-\frac{3}{4} = -0.7500\ldots$; $\frac{1}{3} = 0.333\ldots$; $\sqrt{2} = 1.4142\ldots$

The real numbers $\mathbb{R}$ can be represented as points on the real line:



- three fundamental properties of real numbers
  - algebraic: formalisation of rules of calculation (addition, subtraction, multiplication, division)
    example: $2(3+5) = 2 \cdot 3 + 2 \cdot 5 = 6 + 10 = 16$
  - order: inequalities (geometric picture: see the real line!)
    example: $-\frac{3}{4} < \frac{1}{3} \quad \Rightarrow -\frac{1}{3} < \frac{3}{4}$
  - completeness: there are "no gaps" on the real line

Subsets of the real numbers $\mathbb{R}$

3. Completeness property can be understood by the following construction of the real numbers: (! using set notation !)

Start with "counting numbers" $1, 2, 3, \ldots$

- $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ natural numbers

  $\rightarrow$ can we solve $a + x = b$ for x?
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ integers

  $\rightarrow$ can we solve $ax = b$ for x?
- $\mathbb{Q} = \{\frac{p}{q} | p, q \in \mathbb{Z},\ q \neq 0\}$ rational numbers

  $\rightarrow$ can we solve $x^2 = 2$ for x?
- $\mathbb{R}$ real numbers

  example: positive solution to the equation $x^2 = 2$ is $\sqrt{2}$
  This is an irrational number whose decimal representation is not eventually repeating: $\sqrt{2} = 1.414\ldots$ Another example is $\pi = 3.141\ldots$ which is a Transcendental number

$\mathbb{Q}$ has "holes"

In fact, one has to "prove" this:

Theorem

$x^2 = 2$ *has no solution* $x \in \mathbb{Q}$

The real numbers $\mathbb{R}$ are complete in the sense that they correspond to all points on the real line, i.e., there are no "holes" or "gaps", whereas the rationals have "holes" (namely the irrationals) and

$$\Rightarrow \quad \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$$

Intervals

Definition

A subset of the real line is called an interval if it contains at least two numbers and all the real numbers between any two of its elements.

examples:

- $x > -2$ defines an *infinite interval*; geometrically, it corresponds to a *ray* on the real line
- $3 \leq x \leq 6$ defines a *finite interval*; geometrically, it corresponds to a *line segment* on the real line

So we can distinguish between two basic types of intervals – let's further classify:

Types of Intervals

TABLE 1.1 Types of intervals

	Notation	Set description	Type	Picture
Finite:	(a, b)	$\{x \mid a < x < b\}$	Open	
	$[a, b]$	$\{x \mid a \leq x \leq b\}$	Closed	
	$[a, b)$	$\{x \mid a \leq x < b\}$	Half-open	
	$(a, b]$	$\{x \mid a < x \leq b\}$	Half-open	
Infinite:	(a, ∞)	$\{x \mid x > a\}$	Open	
	$[a, \infty)$	$\{x \mid x \geq a\}$	Closed	
	$(-\infty, b)$	$\{x \mid x < b\}$	Open	
	$(-\infty, b]$	$\{x \mid x \leq b\}$	Closed	
	$(-\infty, \infty)$	$\mathbb{R}$ (set of all real numbers)	Both open and closed	

Finding intervals of numbers

Solve inequalities to find intervals of $x \in \mathbb{R}$:

$$(a)\quad \begin{aligned} 2x - 1 &< x + 3 \\ 2x &< x + 4 \\ x &< 4 \end{aligned}$$

$$(b)\quad \begin{aligned} -\frac{x}{3} &< 2x + 1 \\ -x &< 6x + 3 \\ -\frac{3}{7} &< x \end{aligned}$$

solution sets on the real line:

(a)

(b)

Absolute Value

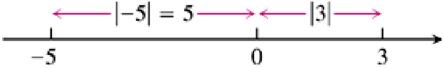
Definition

The absolute value (or *modulus*) of a real number x is

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases} .$$

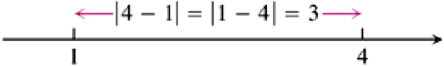
geometrically, $|x|$ is the *distance* between x and 0

example:



$|x - y|$ is the distance between x and y

example:



an alternative definition of $|x|$ is

$$|x| = \sqrt{x^2} \quad ,$$

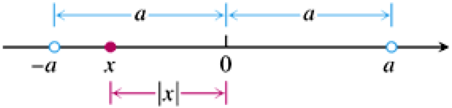
since taking the square root always gives a *non-negative* result!

Inequalities with $|x|$

$|x|$ in an inequality:

$$|x| < a \quad \Leftrightarrow \quad -a < x < a$$

distance from x to 0 is less than $a > 0 \Leftrightarrow x$ must lie between a and $-a$



absolute value properties:

1. $|-a| = |a|$
2. $|ab| = |a|\,|b|$
3. $|\frac{a}{b}| = \frac{|a|}{|b|}$ for $b \neq 0$
4. $|a+b| \leq |a| + |b|$, the *triangle inequality*

prove these statements!

Further properties

Absolute Values and Intervals
If a is any positive number, then

5.	$\lvert x\rvert = a$	if and only if	$x = \pm a$
6.	$\lvert x\rvert < a$	if and only if	$-a < x < a$
7.	$\lvert x\rvert > a$	if and only if	$x > a$ or $x < -a$
8.	$\lvert x\rvert \leq a$	if and only if	$-a \leq x \leq a$
9.	$\lvert x\rvert \geq a$	if and only if	$x \geq a$ or $x \leq -a$

note: "if and only if" is often abbreviated by the sign "$\Leftrightarrow$"

examples

(a) $|2x - 3| \leq 1$

(a)

(b) $|2x - 3| \geq 1$

(b)

Three important inequalities

Triangle inequality

$$|a + b| \leq |a| + |b|$$

arithmetic mean: $\frac{1}{2}(a + b)$; geometric mean $\sqrt{ab}$

Arithmetic-geometric mean inequality

$$\sqrt{ab} \leq \frac{1}{2}(a + b) \qquad \text{for } a, b \geq 0$$

Cauchy-Schwarz inequality

$$(ac + bd)^2 \leq (a^2 + b^2)(c^2 + d^2)$$

Proof of the arithmetic-geometric mean inequality

- multiply inequality by 2 and square:

$$\sqrt{ab} \leq \frac{1}{2}(a+b) \quad \Leftrightarrow 4ab \leq (a+b)^2$$

- use direct proof: start on RHS until the LHS is obtained

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &= a^2 + 2ab + 2ab - 2ab + b^2 \\ &= 4ab + (a-b)^2 \quad , \quad (a-b)^2 \geq 0 \text{ and therefore} \\ &\geq 4ab\end{aligned}$$

What is a function?

examples:

height of the floor of the lecture hall depending on distance; stock market index depending on time; volume of a sphere depending on radius

What do we mean when we say

y is a function of x?

Symbolically, we write $y = f(x)$, where

- x is the independent variable (input value of f)
- y is the dependent variable (output value of f at x)
- f is a function ("rule that assigns x to y")

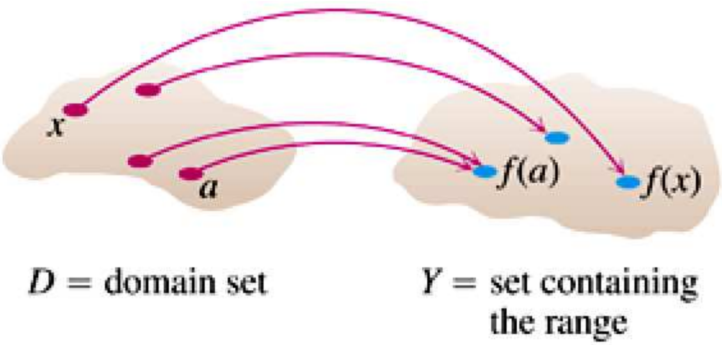
a function acts like a "little machine":

Important: uniqueness – only *one value* $f(x)$ for every x!

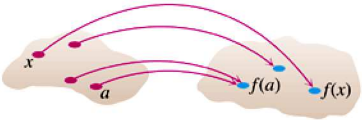
Definition of a function

Definition

A function from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.



Domain, range and some notation



- The set D of all possible *input values* is called the domain of f.
- The set R of all possible *output values* of $f(x)$ as x varies throughout D is called the range of f.

note: $R \subseteq Y$!

- We write f *maps* D *to* Y symbolically as

$$f : D \to Y$$

- We write f *maps* x *to* $y = f(x)$ symbolically as

$$f : x \mapsto y = f(\mathrm{x})$$

Note the different arrow symbols used! (Maplet)

Natural domain

The natural domain is the largest set of real x which the rule f can be applied to.

examples:

Function	**Domain** $x \in D$	**Range** $y \in R$
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

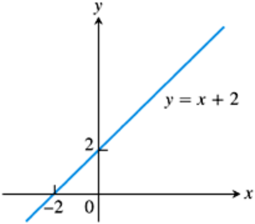
Graphs of functions

Definition

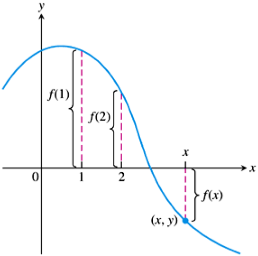
If f is a function with domain D, its graph consists of the points (x, y) whose coordinates are the input-output pairs for f:

$$\{(x, f(x)) | x \in D\}$$

examples:



given the function, one can *sketch* the graph

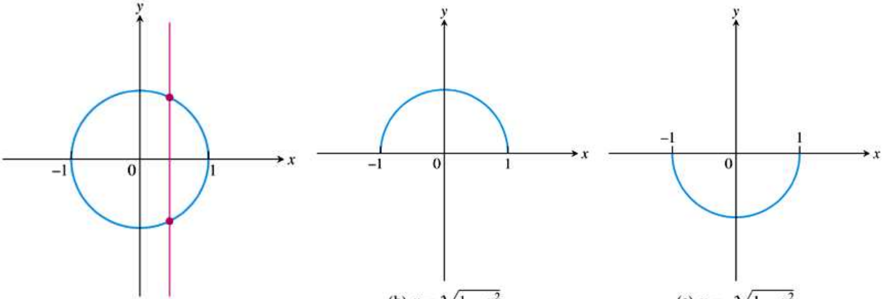


$y = f(x)$ is the *height* of the graph above/below x.

Arbitrary curves vs. graphs of functions

recall: A function f can have only one value $f(x)$ for each x in its domain! This leads to the vertical line test:

No vertical line can intersect the graph of a function *more than once*.



(a) $x^2 + y^2 = 1$

(b) $y = \sqrt{1 - x^2}$

(c) $y = -\sqrt{1 - x^2}$

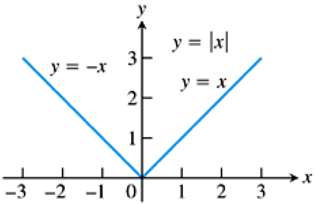
Piecewise defined functions

A piecewise defined function is a function that is described by using different formulas on different parts of its domain.

examples:

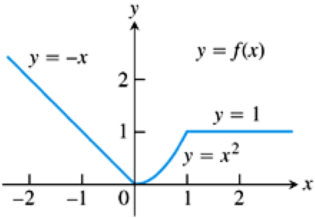
- the *absolute value function*

$$f(x) = |x| = \begin{cases} x & , x \geq 0 \\ -x & , x < 0 \end{cases}$$



- some other function

$$f(x) = \begin{cases} -x & , x < 0 \\ x^2 & , 0 \leq x \leq 1 \\ 1 & , x > 1 \end{cases}$$



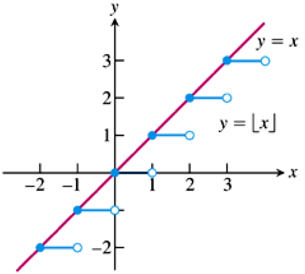
Floor and ceiling functions

- the floor function

$$f(x) = \lfloor x \rfloor$$

is given by the greatest integer less than or equal to x:

$$\lfloor 1.3 \rfloor = 1,\ \lfloor -2.7 \rfloor = -3$$

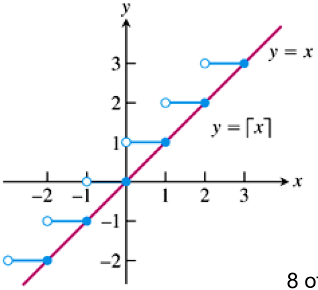


- the ceiling function

$$f(x) = \lceil x \rceil$$

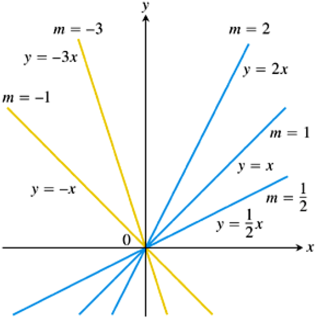
is given by the smallest integer greater than or equal to x:

$$\lceil 3.5 \rceil = 4,\ \lceil -1.8 \rceil = -1$$



Some fundamental types of functions

- linear function $\boxed{f(x) = mx + b}$

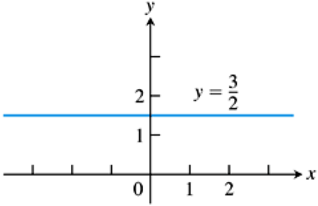


$b = 0$: all lines pass through the origin,

$$f(x) = mx$$

One also says "$y = f(x)$ is proportional to x" for some nonzero constant m.

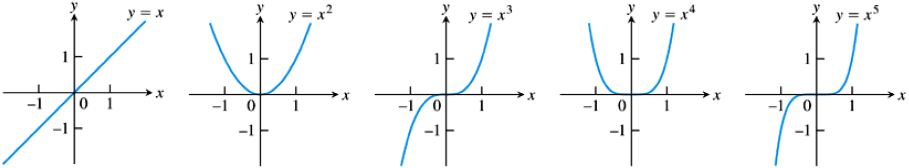
$m = 0$: constant function $\boxed{\text{f(x)=b}}$



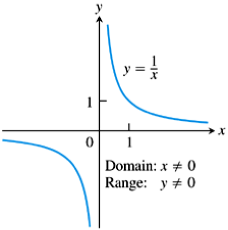
Power function I

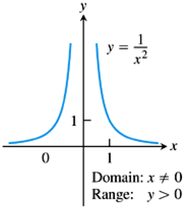
- power function $\boxed{f(x) = x^a}$

  $a = n \in \mathbb{N}$: graphs of $f(x)$ for $n = 1, 2, 3, 4, 5$



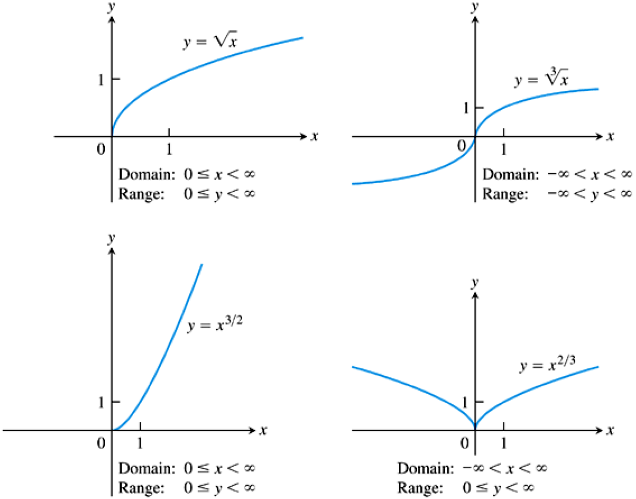
$a = -n$, $n \in \mathbb{N}$: graphs of $f(x)$ for $n = -1, -2$





Power function II

still power function $\boxed{f(x) = x^a}$, now for $a \in \mathbb{Q}$: graphs of $f(x)$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}, \frac{2}{3}$



Polynomials

- polynomials

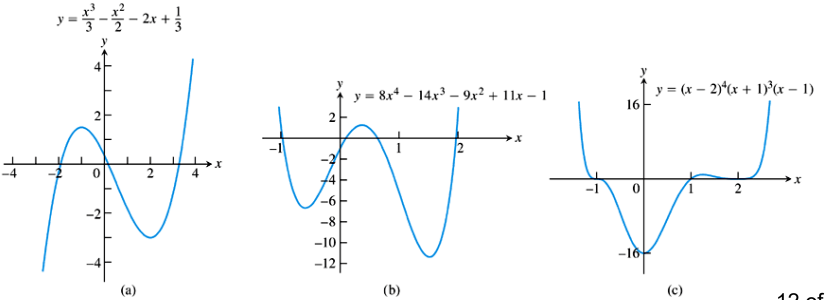
$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\ ,\ \ n \in \mathbb{N}$$

with $a_n \neq 0$, *coefficients* $a_0, a_1, \ldots, a_{n-1}, a_n \in \mathbb{R}$ and domain $\mathbb{R}$

n is called the *degree* of the polynomial

examples: linear functions with $m \neq 0$ are polynomials of degree 1

three polynomial functions and their graphs



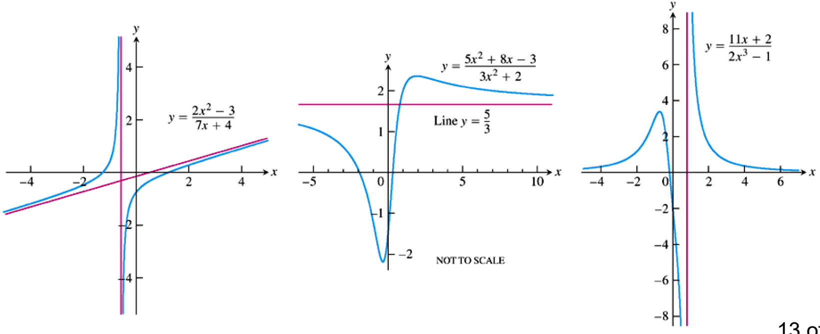
Rational functions

- rational functions

$$\boxed{f(x) = \frac{p(x)}{q(x)}}$$

with $p(x)$ and $q(x)$ polynomials and domain $\mathbb{R} \setminus \{x | q(x) = 0\}$ (never divide by zero!)

examples: three rational functions and their graphs

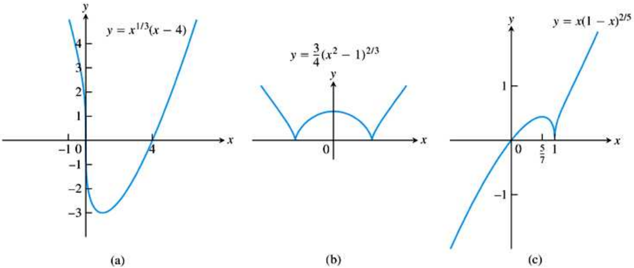


(a) (b) (c)

Even more types of functions

Other classes (to come later):

- algebraic functions



- trigonometric functions
- exponential and logarithmic functions
- ....

Increasing/decreasing functions

Informally,

- a function is called increasing if the graph of the function "climbs" or "rises" as you move *from left to right*.
- a function is called decreasing if the graph of the function "descends" or "falls" as you move *from left to right*.

examples:

function	**where increasing**	**where decreasing**
$y = x^2$	$0 \leq x < \infty$	$-\infty < x \leq 0$
$y = 1/x$	nowhere	$-\infty < x < 0$ and $0 < x < \infty$
$y = 1/x^2$	$-\infty < x < 0$	$0 < x < \infty$
$y = x^{2/3}$	$0 \leq x < \infty$	$-\infty < x \leq 0$

Even/odd functions

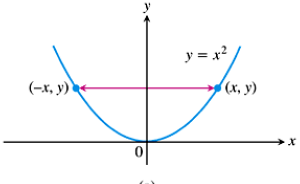
Definition

A function $y = f(x)$ is an

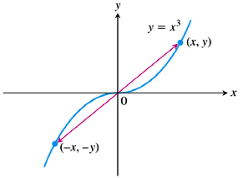
- even function of x if $f(-x) = f(x)$
- odd function of x if $f(-x) = -f(x)$

for every x in the function's domain.

examples:



(a)



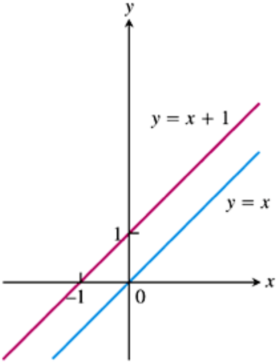
(b)

$f(-x) = (-x)^2 = x^2 = f(x)$: even function; graph is *symmetric about the y-axis*

$f(-x) = (-x)^3 = -x^3 = -f(x)$: odd function; graph is *symmetric about the origin*

Even/odd functions continued

further examples:



1. $f(-x) = -x = -f(x)$: odd function
2. $f(-x) = -x + 1 \neq f(x)$ and $-f(x) = -x - 1 \neq f(-x)$: neither even nor odd!

Sums, differences, products, quotients

If f and g are functions, then for every

$$x \in D(f) \cap D(g)$$

(that is, for every x that belongs to the domains of *both* f and g) we define

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\(f-g)(x) &= f(x)-g(x)\\(fg)(x) &= f(x)g(x)\\(f/g)(x) &= f(x)/g(x) \quad \text{if } g(x) \neq 0\end{aligned}$$

algebraic operation on functions = algebraic operation on function values

Special case: multiplication by a constant $c \in \mathbb{R}$:

$$(cf)(x) = c\,f(x)$$

(take $g(x) = c$ constant function)

Combining functions algebraically

examples:

$$f(x) = \sqrt{x} \quad , \quad g(x) = \sqrt{1-x}$$

(natural) domains:

$$D(f) = [0, \infty) \qquad D(g) = (-\infty, 1]$$

intersection of both domains:

$$D(f) \cap D(g) = [0, \infty) \cap (-\infty, 1] = [0, 1]$$

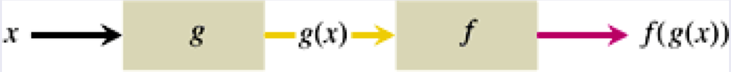
function	**formula**	**domain**
$f+g$	$(f+g)(x) = \sqrt{x} + \sqrt{1-x}$	$[0,1] = D(f) \cap D(g)$
$f-g$	$(f-g)(x) = \sqrt{x} - \sqrt{1-x}$	$[0,1]$
$g-f$	$(g-f)(x) = \sqrt{1-x} - \sqrt{x}$	$[0,1]$
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	$[0,1]$
f/g	$\frac{f}{g}(x) = \frac{f(x)}{g(x)} = \sqrt{\frac{x}{1-x}}$	$[0,1)$ ($x=1$ excluded)
g/f	$\frac{g}{f}(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0,1]$ ($x=0$ excluded)

Composition of functions

Definition

If f and g are functions, the composite function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) = f(g(x))$$

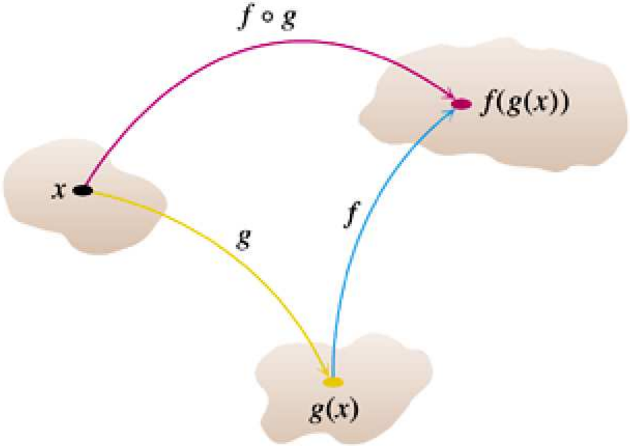


The *domain* of $f \circ g$ consists of the numbers x in the domain of g for which $g(x)$ lies in the domain of f, i.e.

$$D(f \circ g) = \{x | x \in D(g) \text{ and } g(x) \in D(f)\}$$

Arrow diagram for a composite function

$$D(f \circ g) = \{x | x \in D(g) \text{ and } g(x) \in D(f)\}$$



Finding formulas for composites

examples:

$$\begin{aligned} f(x) &= \sqrt{x} \quad \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x + 1 \quad \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	**domain**
$(f \circ g)(x) = f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
$(g \circ f)(x) = g(f(x)) = f(x) + 1 = \sqrt{x} + 1$	$[0, \infty)$
$(f \circ f)(x) = f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$(g \circ g)(x) = g(g(x)) = g(x) + 1 = x + 2$	$(-\infty, \infty)$

The domain of composites

further examples:

$$\begin{aligned} f(x) &= \sqrt{x} \quad \text{with} \quad D(f) = [0, \infty) \\ g(x) &= x^2 \quad \text{with} \quad D(g) = (-\infty, \infty) \end{aligned}$$

composite	domain
$(f \circ g)(x) = \lvert x \rvert$	$(-\infty, \infty)$
$(g \circ f)(x) = x$	$[0, \infty)$

See Thomas Calculus Sections: (1.1, 1.3, 1.4, and 1.5) for more examples

Mathematics (MATH113)

Week 2 (Lectures No. 3&4)

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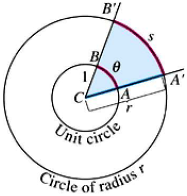
2023

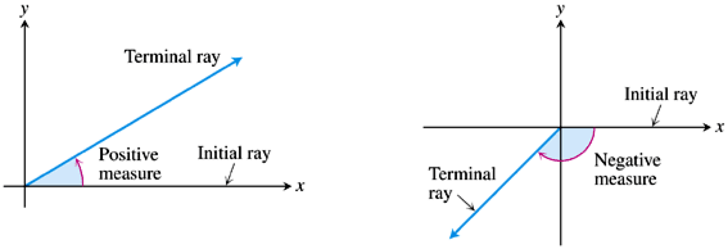
Radian measure



The radian measure of the angle ACB is the length θ of arc AB on the unit circle.
$s = r\theta$ is the length of arc on a circle of radius r when θ is measured in radians.
conversion formula degrees $\leftrightarrow$ radians:

$$360^\circ \text{ corresponds to } 2\pi \;\Rightarrow\; \boxed{\frac{\text{angle in radians}}{\text{angle in degrees}} = \frac{\pi}{180}}$$

Signed angles

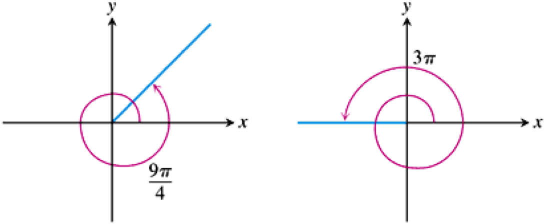


- angles are oriented
- positive angle: counter-clockwise
- negative angle: clockwise

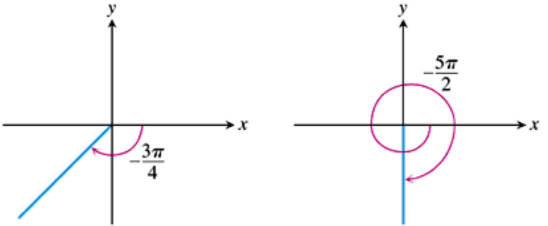
Large angles

note: angles can be larger than 2π:

counter-clockwise:

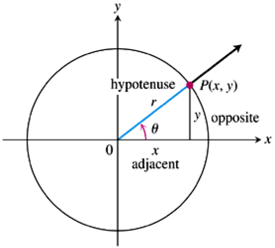


clockwise:



Trigonometric functions

reminder: the six basic trigonometric functions

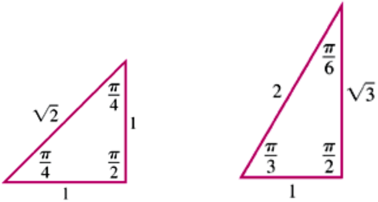


sine:	$\sin\theta = \frac{y}{r}$	**cosecant:**	$\csc\theta = \frac{r}{y}$
cosine:	$\cos\theta = \frac{x}{r}$	**secant:**	$\sec\theta = \frac{r}{x}$
tangent:	$\tan\theta = \frac{y}{x}$	**cotangent:**	$\cot\theta = \frac{x}{y}$

note: These definitions hold not only for $0 \leq \theta \leq \pi$ but also for $\theta < 0$ and $\theta > \pi/2$.

Finding trigonometric function values

recommended to memorize the following two triangles:



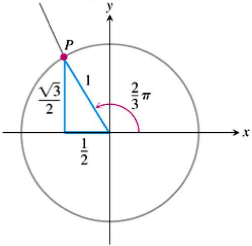
because exact values of trigonometric ratios can be read from them

example:

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \quad ; \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Finding extended trigonometric function values

a more non-trivial **example:**



$\sin \frac{2}{3}\pi = \frac{y}{r} = \sin\left(\pi - \frac{2}{3}\pi\right) = \sin \frac{\pi}{3}$

see previous triangle: $\sin \frac{\pi}{3} = \sqrt{3}/2$

here $r = 1 \Rightarrow x = -1/2\,,\; y = \sqrt{3}/2$ (why?)

from the above triangle we can now read off the values of all trigonometric functions:

$$\sin\left(\frac{2}{3}\pi\right) = \frac{y}{r} = \frac{\sqrt{3}}{2} \qquad \csc\left(\frac{2}{3}\pi\right) = \frac{r}{y} = \frac{2}{\sqrt{3}}$$

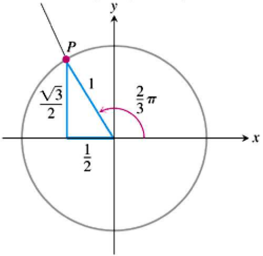
$$\cos\left(\frac{2}{3}\pi\right) = \frac{x}{r} = -\frac{1}{2} \qquad \sec\left(\tfrac{2}{3}\pi\right) = \tfrac{r}{x} = -2$$

$$\tan\left(\frac{2}{3}\pi\right) = \frac{y}{x} = -\sqrt{3} \qquad \cot\left(\frac{2}{3}\pi\right) = \frac{x}{y} = -\frac{1}{\sqrt{3}}$$

Periodic functions

note: for angle of measure θ and angle of measure $\theta + 2\pi$ we have the *very same* trigonometric function values

example:



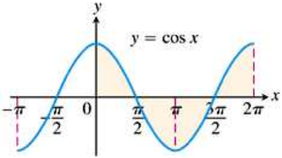
$$\sin(\theta + 2\pi) = \sin\theta$$
$$\cos(\theta + 2\pi) = \cos\theta$$
$$\tan(\theta + 2\pi) = \tan\theta$$

and so on

DEFINITION Periodic Function

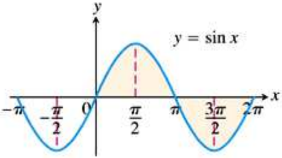
A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Graphs of trigonometric functions



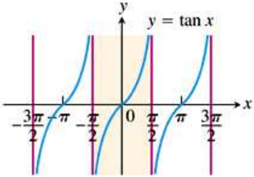
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(a)



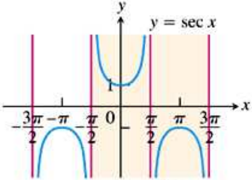
Domain: $-\infty < x < \infty$
Range: $-1 \le y \le 1$
Period: 2π

(b)



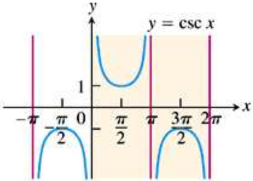
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $-\infty < y < \infty$
Period: π

(c)



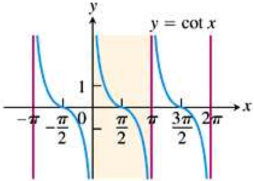
Domain: $x \neq \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

(d)



Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $y \le -1$ and $y \ge 1$
Period: 2π

(e)

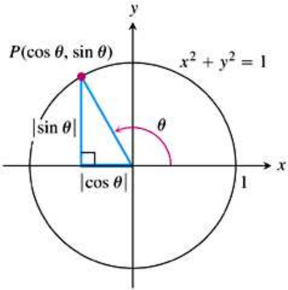


Domain: $x \neq 0, \pm\pi, \pm 2\pi, \ldots$
Range: $-\infty < y < \infty$
Period: π

(f)

An important trigonometric identity

Since $x = r\cos\theta$ and $y = r\sin\theta$ by definition, for a triangle with $r = 1$ we immediately have $\boxed{\cos^2\theta + \sin^2\theta = 1}$ (why?)



This is an example of an identity, i.e., an equation that remains true *regardless of the values of any variables that appear within it.*

counterexample:

$$\cos\theta = 1$$

This is *not* an identity, because it is only true for *some* values of θ, not all.

Important trigonometric formulas

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$1 + cot^2(\theta) = \csc^2(\theta)$$

$$\cos(A \mp B) = \cos(A) cos(B) \pm \sin(A) sin(B)$$

$$\sin(A \mp B) = \sin(A) cos(B) \mp cos(A) \sin(B)$$

Important trigonometric formulas

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

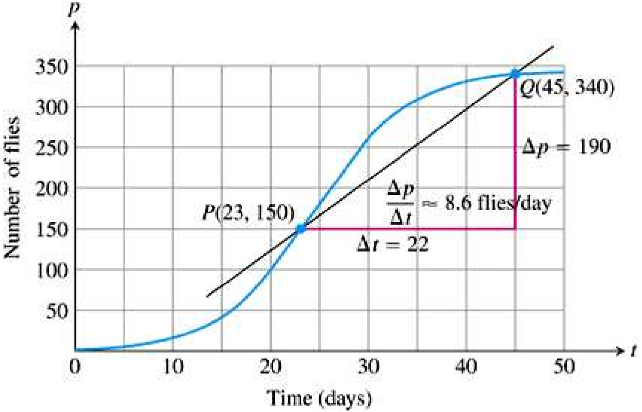
$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{2}$$

See Thomas Calculus section 1.6

Average rate of change

example: growth of a fruit fly population measured experimentally

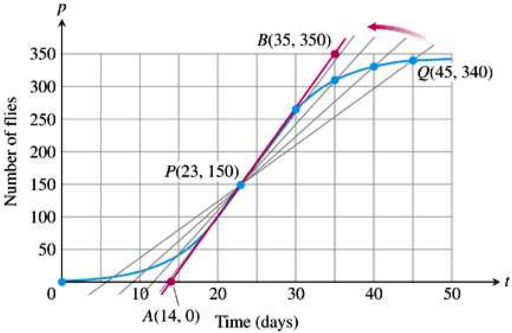


- average rate of change from day 23 to day 45?
- growth rate on day a specific day, e.g., day 23?

Growth rate on a specific day

study the average rates of change over increasingly short time intervals starting at day 23:

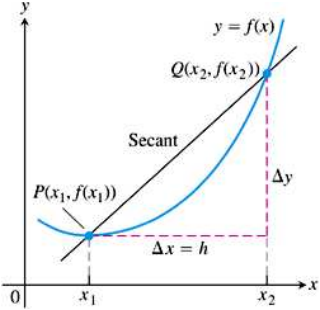
Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$



lines approach the red tangent at point P with slope

$$\frac{350-0}{35-14} \simeq 16.7 \text{ flies/day}$$

Summary: average rate of change and limit



$$x_2 = x_1 + h$$

DEFINITION Average Rate of Change over an Interval

The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Informal definition of a limit

Definition

Let $f(x)$ be defined on an open interval about x_0 *except possibly at* x_0 *itself*. If $f(x)$ gets arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to x_0, we say that f approaches the limit L as x approaches x_0, and we write

$$\lim_{x \longrightarrow x_0} f(x) = L \, ,$$

Behaviour of a function near a point

example: How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x_0 = 1$?

- problem: $f(x)$ is not defined for $x_0 = 1$
- but: we can *simplify* for $x \neq 1$:

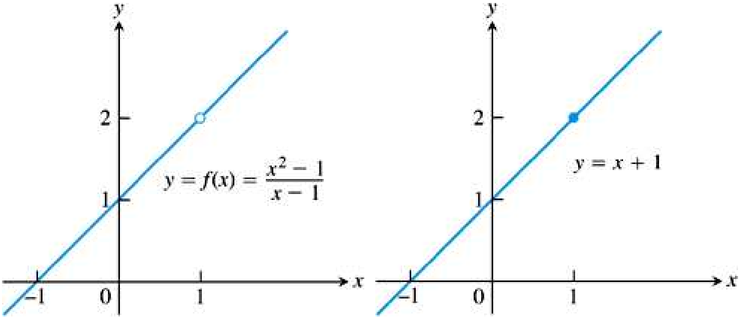
$$f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1 \text{ for } x \neq 1$$

- this *suggests* that

$$\lim_{x \to 1} f(x) = 1 + 1 = 2$$

Limit: a geometric view

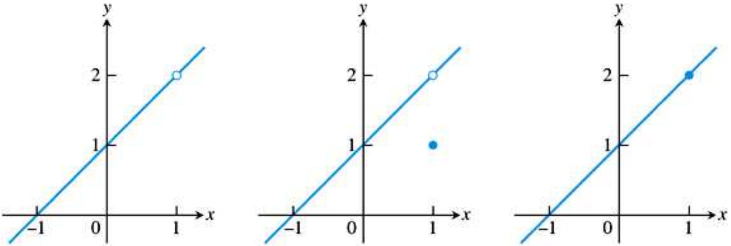
graphs of these two functions:



We say that $f(x)$ approaches the limit 2 as x approaches 1 and write

$$\lim_{x \to 1} f(x) = 2$$

The limit value does not depend on how the function is defined at x_0



(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$

(b) $g(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$

(c) $h(x) = x + 1$

All these functions have limit 2 as $x \to 1$!

However, only for h we have equality of limit and function value:

$$\lim_{x \to 1} h(x) = h(1)$$

Finding limits of simple functions

We have just "convinced ourselves" that for real constants k and c

$$\lim_{x \longrightarrow c} x = c$$

and

$$\lim_{x \longrightarrow c} k = k \quad .$$

The following important theorem provides the basis to calculate limits of functions that are arithmetic combinations of the above two functions (like polynomials, rational functions, powers):

Limit laws

Theorem

If L, M, c and k are real numbers and

$$\lim_{x \to c} f(x) = L \text{ and } \lim_{x \to c} g(x) = M \text{ , then}$$

1. *Sum Rule:* $\lim_{x \to c}(f(x) + g(x)) = L + M$
   The limit of the sum of two functions is the sum of their limits.
2. *Difference Rule:* $\lim_{x \to c}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x \to c}(f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x \to c}(k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule: If s and r are integers with no common factor and $s \neq 0$, then*

   $$\lim_{x \to c}(f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Using limit laws

. . . concerning proofs of this theorem see later . . .

examples:

- $\lim_{x\to c}(x^3 - 4x + 2) =$ (rules 1,2)
  $= \lim_{x\to c} x^3 - \lim_{x\to c} 4x + \lim_{x\to c} 2 =$ (rules 3 or 6,4)
  $= c^3 - 4c + 2$
- $\displaystyle\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{c^4 + c^2 - 1}{c^2 + 5}$ (rules 5,1,2,3 or 6)
- $\displaystyle\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{4(-2)^2 - 3} = \sqrt{13}$ (rules 6,2, 3 or 6,4)

So "sometimes" you can just *substitute the value of* x.

Eliminating zero denominators algebraically

example: Evaluate

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x}$$

- substitution of $x = 1$? *No!*
- *but* algebraic simplification is possible:

$$\frac{x^2 + x - 2}{x^2 - x} = \frac{(x+2)(x-1)}{x(x-1)} = \frac{x+2}{x}\,, \; x \neq 1$$

- therefore,

$$\lim_{x \to 1} \frac{x^2 + x - 2}{x^2 - x} = \lim_{x \to 1} \frac{x+2}{x} = 3$$

Creating and cancelling a common factor

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2}$$

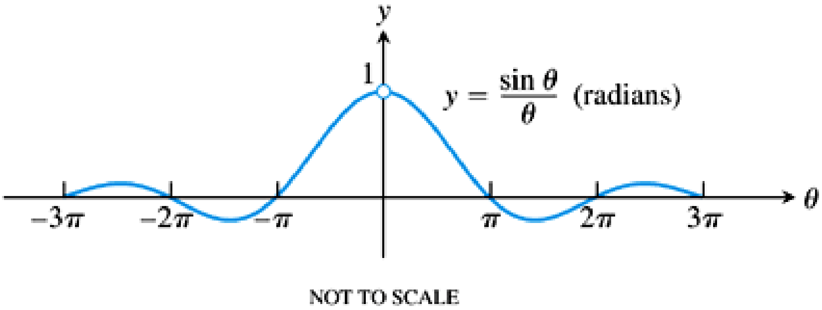
- substitution of $x = 0$?
- trick: algebraic simplification

$$\begin{aligned}
\frac{\sqrt{x^2 + 100} - 10}{x^2} &= \frac{\sqrt{x^2 + 100} - 10}{x^2} \frac{\sqrt{x^2 + 100} + 10}{\sqrt{x^2 + 100} + 10} \\
&= \frac{(x^2 + 100) - 100}{x^2(\sqrt{x^2 + 100} + 10)} \\
&= \frac{1}{\sqrt{x^2 + 100} + 10}
\end{aligned}$$

- therefore

$$\lim_{x \to 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \lim_{x \to 0} \frac{1}{\sqrt{x^2 + 100} + 10} = \frac{1}{20}$$

Limits involving $\sin\theta/\theta$

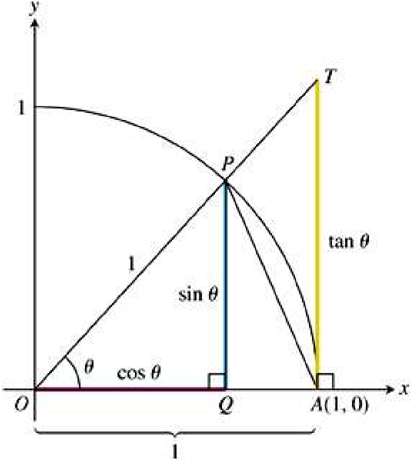


Theorem

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1 \quad (\theta \textit{ in radians})$$

Proof of $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$

show that both right-hand and left-hand limits are equal to 1:



restrict to $0 < \theta < \pi/2$

$$\sin\theta < \theta < \tan\theta$$

proof via areas of two triangles and area sector; this implies

$$\cos\theta < \frac{\sin\theta}{\theta} < 1 \quad .$$

by sandwich theorem (taking the limit as $\theta \to 0^+$)

$$1 \le \lim_{\theta\to 0^+} \frac{\sin\theta}{\theta} \le 1 \quad .$$

symmetry: also $\lim_{\theta\to 0^-} \frac{\sin\theta}{\theta} = 1$

$$\Rightarrow \lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$$

Applications of this theorem

examples:

(1) Compute

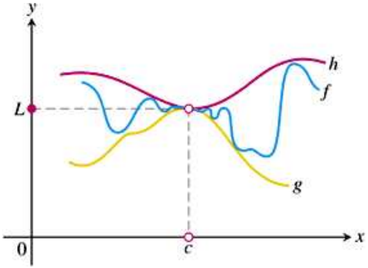
$$\begin{aligned} & \lim_{h\to 0} \frac{\cos h - 1}{h} = && (\sin^2(h/2) = (1-\cos h)/2) \\ = \; & \lim_{h\to 0} \frac{1 - 2\sin^2(h/2) - 1}{h} \\ = \; & \lim_{h\to 0} -\frac{\sin(h/2)}{h/2}\sin(h/2) && (\theta = h/2) \\ = \; & \lim_{\theta\to 0} -\frac{\sin\theta}{\theta}\sin\theta && (\text{limit laws}) \\ = \; & -1\cdot 0 = 0 \end{aligned}$$

Applications of this theorem

(2) Compute

$$\begin{aligned} & \lim_{x\to 0} \frac{\sin 2x}{5x} = \\ = \quad & \lim_{x\to 0} \frac{(2/5)\cdot \sin 2x}{(2/5)\cdot 5x} \\ = \quad & \lim_{x\to 0} \frac{2}{5}\frac{\sin 2x}{2x} \qquad (\theta = 2x) \\ = \quad & \lim_{\theta\to 0} \frac{2}{5}\frac{\sin \theta}{\theta} \qquad \text{(limit laws)} \\ = \quad & \frac{2}{5} \end{aligned}$$

The Sandwich Theorem



function f sandwiched between g and h that have the same limit

THEOREM 4 The Sandwich Theorem

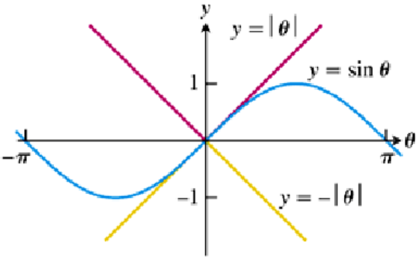
Suppose that $g(x) \leq f(x) \leq h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

Then $\lim_{x \to c} f(x) = L$.

Application

example: Show that $\lim_{\theta \to 0} \sin\theta = 0$.



(a)

- From the definition of $\sin\theta$ it follows that

$$-|\theta| \leq \sin\theta \leq |\theta|$$

- We have

$$\lim_{\theta \to 0}(-|\theta|) = \lim_{\theta \to 0} |\theta| = 0$$

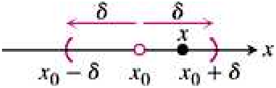
- Using the sandwich theorem, we therefore conclude that

$$\lim_{\theta \to 0} \sin\theta = 0$$

- Similarly, one can prove that $\lim_{\theta \to 0} \cos\theta = 1$

Limits: trying to be more precise

- We have used informal phrases such as "sufficiently close". But what do they mean?
- A picture might help:



- Let's be precise: instead of

"for all x sufficiently close to x_0 ..."

write

"choose $\delta > 0$ such that for all x, $0 < |x - x_0| < \delta$ ..."

The precise definition of a limit

DEFINITION Limit of a Function

Let $f(x)$ be defined on an open interval about x_0, except possibly at x_0 itself. We say that the **limit of $f(x)$ as x approaches x_0 is the number L**, and write

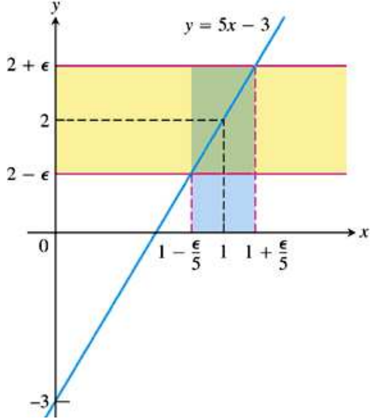
$$\lim_{x \to x_0} f(x) = L,$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all x,

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon.$$

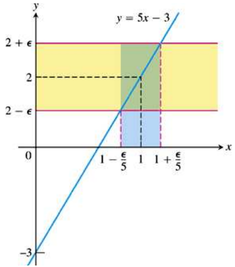
Testing the definition, part 1

example: show that $\lim_{x\to 1}(5x-3) = 2$; graphically:



Testing the definition, part 2

example: show that $\lim_{x\to 1}(5x-3)=2$; algebraically:



- $|f(x)-L|<\epsilon$: this is what we want to be fulfilled!
  substitute: $|(5x-3)-2|<\epsilon$
  $\Leftrightarrow \quad |5x-5|<\epsilon$
  $\Leftrightarrow \boxed{|x-1|<\frac{1}{5}\epsilon}$ (1)
- given this inequality, we now need to find a $\delta>0$ such that
  $0<|x-x_0|<\delta$ is fulfilled
  substitute: $\boxed{0<|x-1|<\delta}$ (2)

- *matching* (1) with (2) suggests to choose $\delta=\frac{1}{5}\epsilon$, because:
  if $0<|x-1|<\delta=\epsilon/5$, then $|f(x)-2|=5|x-1|<5\delta=\epsilon$
  for all ϵ.

General recipe of how to apply the definition

How to Find Algebraically a δ for a Given f, L, x_0, and $\epsilon > 0$

The process of finding a $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad |f(x) - L| < \epsilon$$

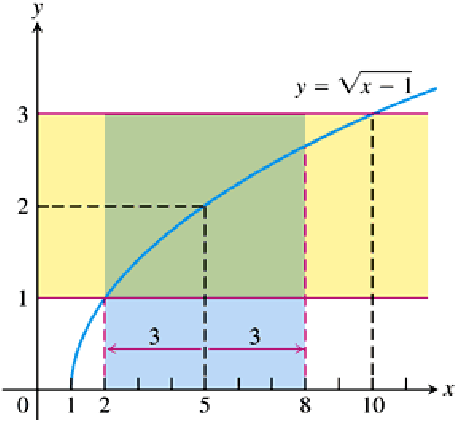
can be accomplished in two steps.

1. *Solve the inequality* $|f(x) - L| < \epsilon$ to find an open interval (a, b) containing x_0 on which the inequality holds for all $x \neq x_0$.
2. *Find a value of* $\delta > 0$ that places the open interval $(x_0 - \delta, x_0 + \delta)$ centered at x_0 inside the interval (a, b). The inequality $|f(x) - L| < \epsilon$ will hold for all $x \neq x_0$ in this δ-interval.

A slightly more complicated example, part 1

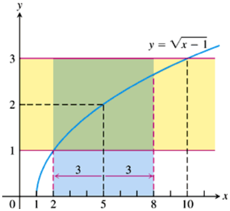
For the limit $\lim_{x\to 5}\sqrt{x-1}=2$ and $\epsilon=1$, find a $\delta>0$ such that for all x

$$0<|x-5|<\delta \Rightarrow |\sqrt{x-1}-2|<1$$



A slightly more complicated example, part 2

Find a $\delta > 0$ such that $|\sqrt{x-1} - 2| < 1$ for all $0 < |x - 5| < \delta$:



1. solve $|f(x) - L| < \epsilon$:
   substitute: $|\sqrt{x-1} - 2| < 1$
   $\Leftrightarrow \quad -1 < \sqrt{x-1} - 2 < 1$
   $\Leftrightarrow \quad 1 < \sqrt{x-1} < 3$
   $\Leftrightarrow \quad 2 < x < 10$
   therefore $(a, b) = (2, 10)$
2. find δ:
   find the distance from $x_0 = 5$ to the *nearest endpoint* of $(2, 10)$, which is $\delta = 3$. Then
   $$x \in (5 - \delta, 5 + \delta) = (2, 8) \subset (2, 10)$$
   means $0 < |x - 5| < 3$, which implies
   $$|\sqrt{x-1} - 2| < 1$$

One-sided limits

- To have a *limit* L as $x \to c$, a function f must be defined on both sides of c (two-sided limit)
- If f fails to have a limit as $x \to c$, it may still have a one-sided limit if the approach is only from the right (*right-hand limit*) or from the left (*left-hand limit*)
- We write

$$\boxed{\lim_{x\to c^+} f(x) = L} \text{ or } \boxed{\lim_{x\to c^-} f(x) = M}$$

- The symbol $x \to c^+$ means that we only consider values of x *greater than* c. The symbol $x \to c^-$ means that we only consider values of x *less than* c.

One-sided limits

- right-hand limit: $\lim_{x \to c^+} f(x) = L$, where $x > c$
- left-hand limit: $\lim_{x \to c^-} f(x) = M$, where $x < c$

THEOREM 6

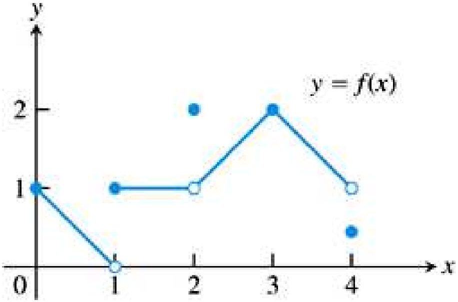
A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x \to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x \to c^-} f(x) = L \quad \text{and} \quad \lim_{x \to c^+} f(x) = L.$$

Limit laws, theorems for limits of polynomials and rational functions, and the sandwich theorem all hold for one-sided limits

Limits of some piecewise linear function

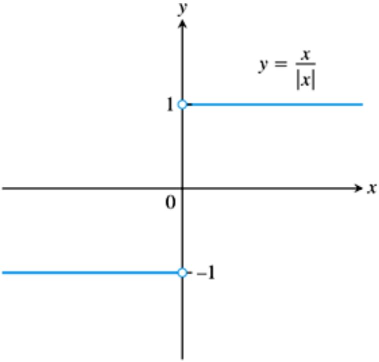
example:



c	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$
0	n.a.	1	n.a.
1	0	1	n.a.
2	1	1	1
3	2	2	2
4	1	n.a.	n.a.

Jump function

example:



- $\lim_{x \to 0^+} f(x) = 1$
- $\lim_{x \to 0^-} f(x) = -1$
- $\lim_{x \to 0} f(x)$
  does not exist

Mathematics (MATH113)

Week 3 (Lectures No. 5 & 6)

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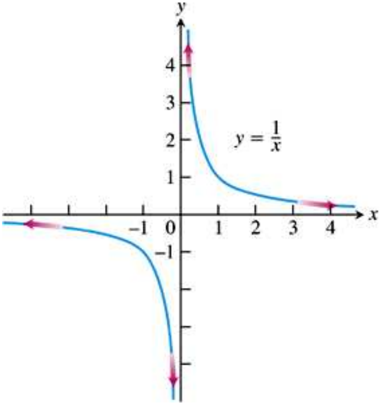
2023

Limits as x approaches infinity

special case of a limit:

x approaching positive/negative infinity

example:



- similar to *one-sided limit*
- use slightly modified ϵ-δ definition of a limit to capture these cases
- **idea** for this: choose a particular δ-*interval* ...

Limits as x approaches infinity: definition

Definition

1. We say that $f(x)$ has the **limit** L **as** x **approaches infinity** and write

$$\lim_{x \to \infty} f(x) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number M such that for all x

$$x > M \quad \Rightarrow \quad |f(x) - L| < \epsilon \quad .$$

2. We say that $f(x)$ has the **limit** L **as** x **approaches minus infinity** and write

$$\lim_{x \to -\infty} f(x) = L$$

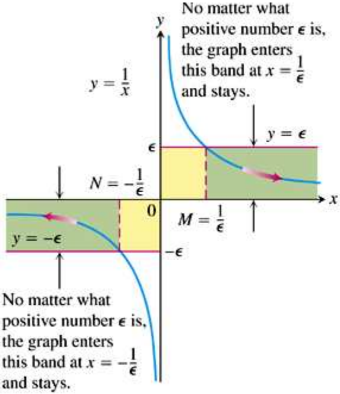
if, for every number $\epsilon > 0$, there exists a corresponding number N such that for all x

$$x < N \quad \Rightarrow \quad |f(x) - L| < \epsilon \quad .$$

Limits at infinity for $f(x) = 1/x$

example:
Show that

$$\lim_{x\to\infty} \frac{1}{x} = 0$$



Let $\epsilon > 0$ be given. We must find a number M such that for all x

$$x > M \Rightarrow \left|\frac{1}{x} - 0\right| = \left|\frac{1}{x}\right| < \epsilon$$

This holds if we choose $M = 1/\epsilon$ or any larger positive number.
(similarly, proof of $\lim_{x\to-\infty} \frac{1}{x} = 0$ and $\lim_{x\to\pm\infty} k = k$)

Limit laws as x approaches infinity

simply replace $x \to c$ by $x \to \pm\infty$ in the previous limit laws theorem:

Theorem

If L, M and k are real numbers and

$$\lim_{x\to\pm\infty} f(x) = L \text{ and } \lim_{x\to\pm\infty} g(x) = M \text{ , then}$$

1. *Sum Rule:* $\lim_{x\to\pm\infty}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to\pm\infty}(f(x) - g(x)) = L - M$
3. *Product Rule:* $\lim_{x\to\pm\infty}(f(x) \cdot g(x)) = L \cdot M$
4. *Constant Multiple Rule:* $\lim_{x\to\pm\infty}(k \cdot f(x)) = k \cdot L$
5. *Quotient Rule:* $\lim_{x\to\pm\infty} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$
6. *Power Rule: If s and r are integers with no common factor and $s \neq 0$, then*

   $$\lim_{x\to\pm\infty} (f(x))^{r/s} = L^{r/s}$$

   provided that $L^{r/s}$ is a real number. (If s is even, we assume that $L > 0$.)

Calculating limits as x approaches infinity

examples: (1)

$$\lim_{x\to\infty}\left(5+\frac{1}{x}\right) = \qquad \text{(sum rule)}$$

$$= \lim_{x\to\infty} 5 + \lim_{x\to\infty}\frac{1}{x} = \qquad \text{(known results)}$$

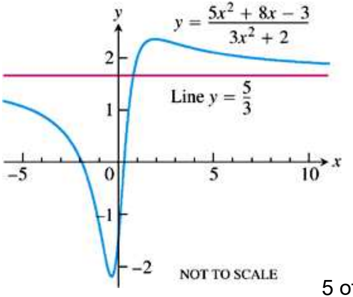
$$= 5$$

(2) method for rationals: pull out highest power of x

$$\lim_{x\to\infty}\frac{5x^2+8x-3}{3x^2+2} =$$

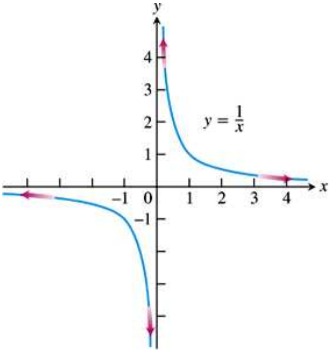
$$= \lim_{x\to\infty}\frac{x^2(5+8/x-3/x^2)}{x^2(3+2/x^2)}$$

$$= \frac{5}{3}$$



Horizontal asymptotes

example:



$$\lim_{x \to \infty} \frac{1}{x} = 0$$

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

The graph approaches the line

$$y = 0$$

asymptotically: the line is an asymptote of the graph.

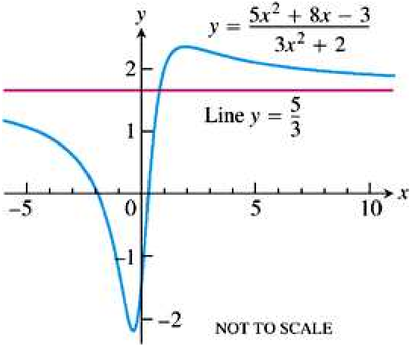
DEFINITION Horizontal Asymptote

A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

Calculating a horizontal asymptote

example: (already seen before)

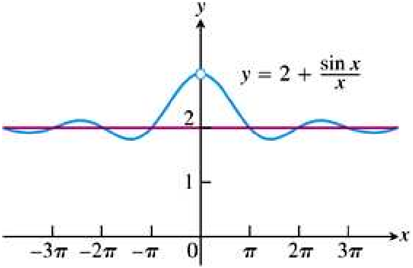


The graph has the line $y = 5/3$ as a horizontal asymptote on *both the left and the right*, because

$$\lim_{x \to \pm\infty} f(x) = \frac{5}{3} \quad .$$

Another application of the sandwich theorem. . .

. . . which also holds for limits such as $x \to \pm\infty$:
Find the horizontal asymptote of $f(x) = 2 + \frac{\sin x}{x}$.



- $0 \leq \left|\frac{\sin x}{x}\right| \leq \left|\frac{1}{x}\right|$ (why?)
- $\lim_{x\to\pm\infty} \left|\frac{1}{x}\right| = 0$
- therefore, by the sandwich theorem,

$$\lim_{x\to\pm\infty} \frac{\sin x}{x} = 0$$

- hence,

$$\lim_{x\to\pm\infty} \left(2 + \frac{\sin x}{x}\right) = 2$$

oblique asymptote

If for a rational function $f(x) = p(x)/q(x)$ the degree of $p(x)$ is *one greater* than the degree of $q(x)$, polynomial division gives

$$f(x) = ax + b + r(x) \qquad \text{with} \lim_{x\to\pm\infty} r(x) = 0$$

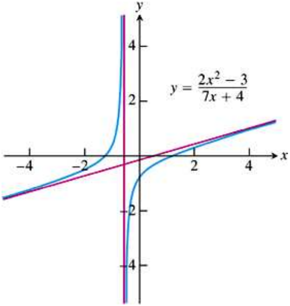
$y = ax + b$ is called an oblique (slanted) asymptote.

example: $f(x) = \dfrac{2x^2-3}{7x+4} = \dfrac{2}{7}x - \dfrac{8}{49} + \dfrac{-115}{49(7x+4)}$

$$\lim_{x\to\pm\infty} \frac{-115}{49(7x+4)} = 0, \text{ so that}$$

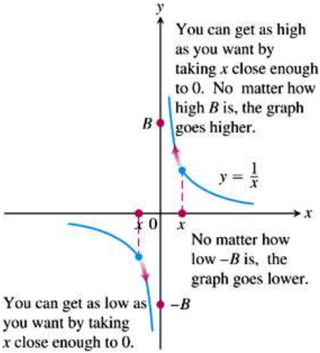
$$y = \frac{2}{7}x - \frac{8}{49}$$

is the oblique asymptote of $f(x)$.



Infinite limits

example:



$f(x) = \frac{1}{x}$ has *no limit* as $x \to 0^+$. However, it is convenient to still say that $f(x)$ *approaches* ∞ as $x \to 0^+$. We write

$$\lim_{x\to 0^+} \frac{1}{x} = \infty$$

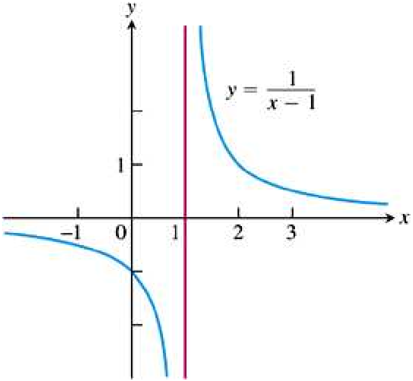
Similarly,

$$\lim_{x\to 0^-} \frac{1}{x} = -\infty$$

note: $\lim_{x\to 0^+} \frac{1}{x} = \infty$ really means that the limit does not exist because $1/x$ becomes arbitrarily large and positive as $x \to 0^+$!

One-sided infinite limits

example: find $\lim_{x\to1^+}\frac{1}{x-1}$ and $\lim_{x\to1^-}\frac{1}{x-1}$



$$\lim_{x\to1^+}\frac{1}{x-1} = \infty$$

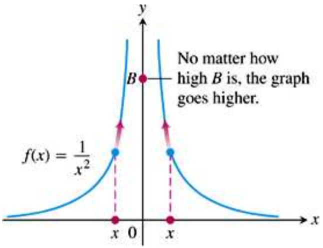
and

$$\lim_{x\to1^-}\frac{1}{x-1} = -\infty$$

as $y = 1/(x-1)$ is just $y = 1/x$ shifted by one to the right.

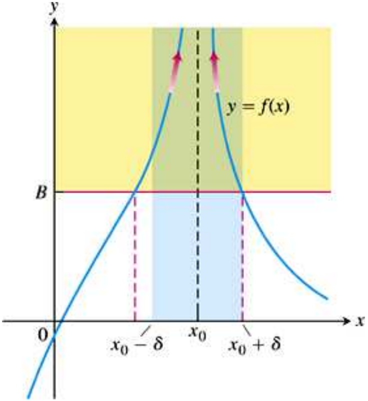
Two-sided infinite limits

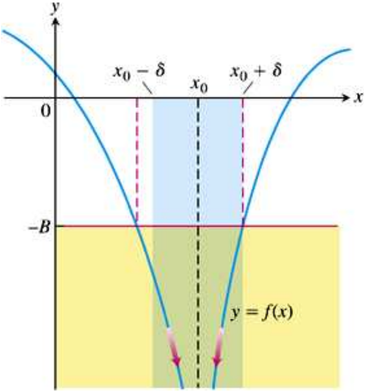
example: what is the behaviour of $f(x) = 1/x^2$ near $x = 0$?



$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

as the values of $1/x^2$ are positive and become arbitrarily large as $x \to 0$.





For $|x - x_0| < \delta$, the graph of $f(x)$ lies

above the line $y = B$ below the line $y = -B$

Precise definition of infinite limits

Definition

1. We say that $f(x)$ **approaches infinity as** x **approaches** x_0 and write

$$\lim_{x \longrightarrow x_0} f(x) = \infty$$

if, for every positive real number B, there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) > B \quad .$$

2. We say that $f(x)$ **approaches negative infinity as** x **approaches** x_0 and write

$$\lim_{x \longrightarrow x_0} f(x) = -\infty$$

if, for every negative real number $-B$, there exists a corresponding $\delta > 0$ such that for all x

$$0 < |x - x_0| < \delta \quad \Rightarrow \quad f(x) < -B \quad .$$

Using the definition

Prove that

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

- given $B>0$, find $\delta>0$ such that

$$0<|x-0|<\delta \quad\Rightarrow\quad \frac{1}{x^2}>B \quad ,$$

  where the last inequality is equivalent to $|x|<1/\sqrt{B}$. Therefore,
- choose $\delta=\frac{1}{\sqrt{B}}$ so that

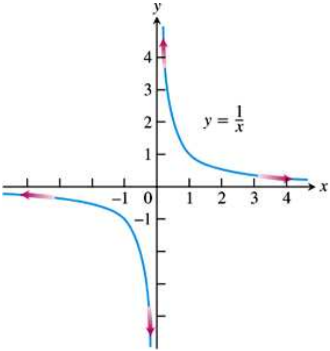
$$0<|x|<\delta\Rightarrow\frac{1}{|x|}>\frac{1}{\delta}\Rightarrow\frac{1}{x^2}>\frac{1}{\delta^2}=B$$

- Hence, by definition

$$\lim_{x\to 0}\frac{1}{x^2}=\infty$$

Vertical asymptotes

example:



$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The graph approaches the line

$$x = 0$$

asymptotically; the line is an asymptote of the graph.

DEFINITION Vertical Asymptote

A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty.$$

An asymptote that is not two-sided

example: Find the horizontal and vertical asymptotes of

$$f(x) = -\frac{8}{x^2 - 4}$$

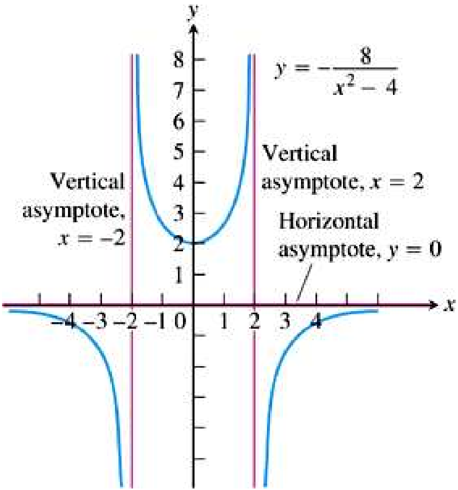
Check for the behaviour as $x \to \pm\infty$ and as $x \to \pm 2$ (why?):

- $\lim_{x\to\pm\infty} f(x) = 0$, approached from *below*.
- $\lim_{x\to-2^-} f(x) = -\infty$, $\lim_{x\to-2^+} f(x) = \infty$
- $\lim_{x\to2^-} f(x) = \infty$, $\lim_{x\to2^+} f(x) = -\infty$ (because $f(x)$ is *even*)

Asymptotes are

$$\boxed{y = 0\,, \quad x = -2\,, \quad x = 2}$$

A one-sided asymptote



curve approaches the x-axis from *only one side*

example: Find the asymptotes of

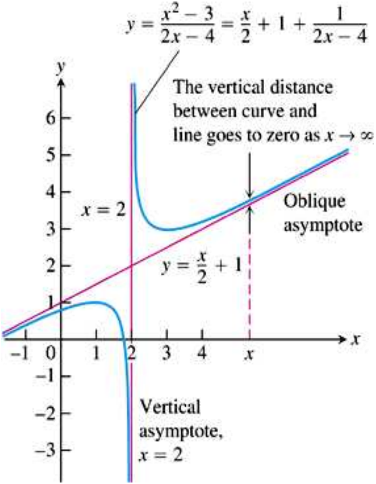
$$f(x) = \frac{x^2 - 3}{2x - 4}$$

- Rewrite by **polynomial division:**

$$f(x) = \frac{x}{2} + 1 + \frac{1}{2x - 4}$$

- Asymptotes are

$$\boxed{y = \tfrac{x}{2} + 1\,, \quad x = 2}$$



We say that $x/2 + 1$ dominates when x is large and that $1/(2x - 4)$ dominates when x is near 2.

.L'Hopital's Rule

Suppose that f(a) = g(a) = 0
f and g are differentiable on an open interval *I* containing a
and that g'(x) ≠ 0 on *I* if x'≠ a. Then

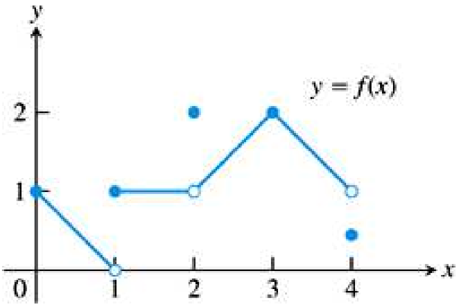
$$\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(a)}{g'(a)}$$

Intuitive approach towards continuity

Definition (informal)

Any function whose graph can be sketched over its domain in one continuous motion, i.e. *without lifting the pen*, is an example of a continuous function.

example:

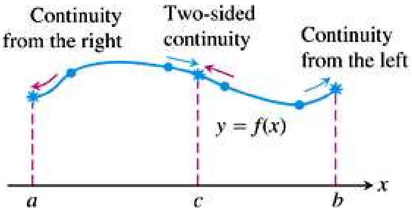


.This function is continuous on $[0, 4]$ *except at* $x = 1$, $x = 2$ and $x = 4$

Continuity at a point

More precisely, we need to define continuity at *interior* and at *end points*.

example:



DEFINITION Continuous at a Point

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \quad \text{or} \quad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively}.$$

Continuity at an interior point

For any $x = c$ in the domain of f one defines:

- right-continuous: $\lim_{x \to c^+} f(x) = f(c)$
- left-continuous: $\lim_{x \to c^-} f(x) = f(c)$

A function f is *continuous at an interior point* $x = c$ if and only if it is *both right-continuous and left-continuous at* c.

Continuity Test

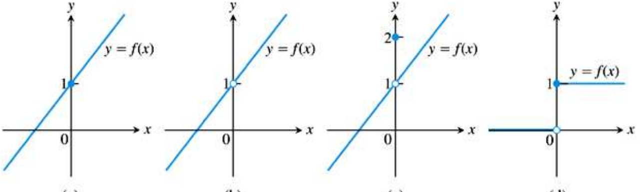
A function $f(x)$ is continuous at $x = c$ if and only if it meets the following three conditions:

1. $f(c)$ exists.
2. f has a limit as x approaches c.
3. The limit equals the function value.

A catalogue of discontinuity types

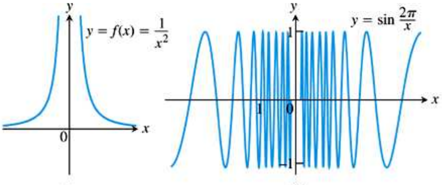
If a function f is not continuous at a point c, we say that f is discontinuous at c. Note that c need not be in the domain of f.

examples:



(a) (b) (c) (d)

continuous not continuous jump discontinuity



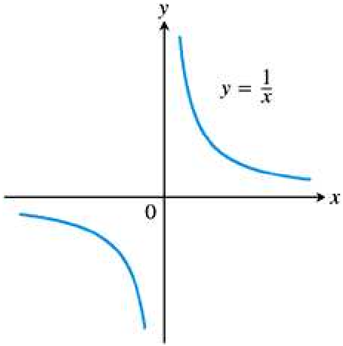
(e) (f)

infinite discontinuity oscillating discontinuity

Continuous functions

- A function is continuous on an interval if and only if it is continuous at every point of the interval.
- A continuous function is a function that is continuous at every point of its domain.

example:



- $y = 1/x$ is a continuous function: It is continuous at every point of its domain.
- It has nevertheless a *discontinuity* at $x = 0$: No contradiction, because it is not defined there.

Algebraic combinations of continuous functions

Previous limit laws straightforwardly imply:

THEOREM 9 Properties of Continuous Functions

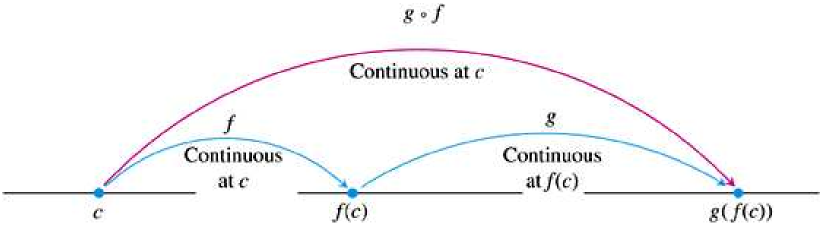
If the functions f and g are continuous at $x = c$, then the following combinations are continuous at $x = c$.

1. *Sums*: $f + g$
2. *Differences*: $f - g$
3. *Products*: $f \cdot g$
4. *Constant multiples*: $k \cdot f$, for any number k
5. *Quotients*: f/g provided $g(c) \neq 0$
6. *Powers*: $f^{r/s}$, provided it is defined on an open interval containing c, where r and s are integers

example: $f(x) = x$ and constant functions are continuous $\Rightarrow$ polynomials and rational functions are also continuous

Continuity for composites

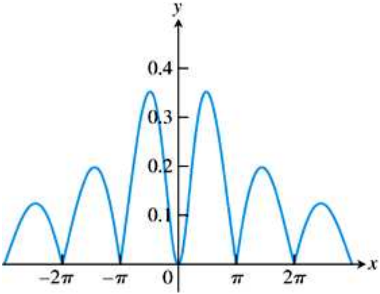
THEOREM 10 Composite of Continuous Functions
If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.



Applying the previous two theorems

Note that $y = \sin x$ and $y = \cos x$ are everywhere continuous:
Show that $y = \left|\frac{x \sin x}{x^2+2}\right|$ is everywhere continuous.

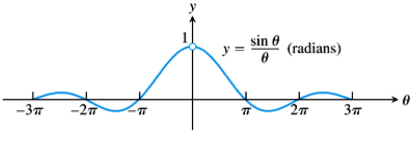
- $f(x) = \frac{x \sin x}{x^2+2}$ is continuous (why?)
- $g(x) = |x|$ is continuous (why?)
- therefore $y = g \circ f(x)$ is continuous



Continuous extension to a point

example:

$$f(x) = \frac{\sin x}{x}$$



is defined and continuous for all $x \neq 0$. As $\lim_{x \to 0} \frac{\sin x}{x} = 1$, it makes sense to *define a new function*

$$F(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0 \end{cases}$$

Definition

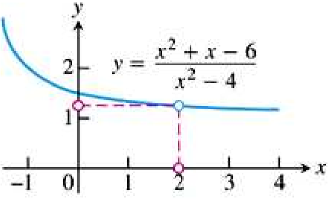
If $\lim_{x \to c} f(x) = L$ exists, but $f(c)$ is not defined, we define a new function

$$F(x) = \begin{cases} f(x) & \text{for } x \neq c \\ L & \text{for } x = c \end{cases},$$

which is continuous at c. It is called the continuous extension of $f(x)$ to c.

Finding continuous extensions

example: Find the continuous extension of $f(x) = \dfrac{x^2 + x - 6}{x^2 - 4}$ to $x = 2$.



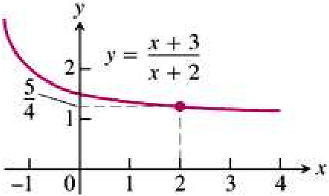
(a)

For $x \neq 2$, $f(x)$ is equal to

$$F(x) = \frac{x+3}{x+2} \text{ (why?)}$$

$F(x)$ is the continuous extension of $f(x)$ to $x = 2$, as

$$\lim_{x \to 2} f(x) = \frac{5}{4} = F(2)$$

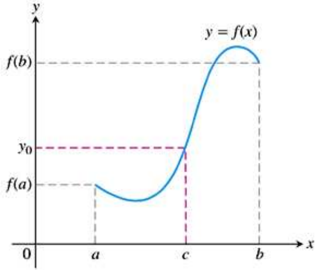


(b)

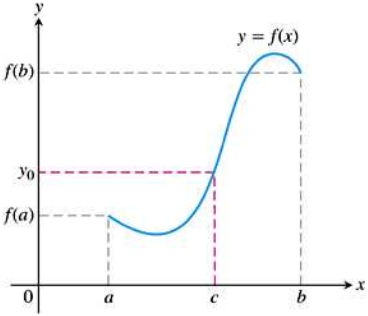
The intermediate value theorem

A function has the intermediate value property if whenever it takes on two values, it also takes on all the values in between.

THEOREM 11 The Intermediate Value Theorem for Continuous Functions
A function $y = f(x)$ that is continuous on a closed interval $[a, b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



Geometrical interpretation of this theorem



- Any horizontal line crossing the y-axis between $f(a)$ and $f(b)$ will cross the curve $y = f(x)$ at least once over the interval $[a, b]$.
- Continuity is essential: if f is discontinuous at any point of the interval, then the function may "jump" and miss some values.

Mathematics (MATH113)

Week 4 (Lectures No. 7 & 8)

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calculate slope and tangent

Finding the Tangent to the Curve $y = f(x)$ at (x_0, y_0)

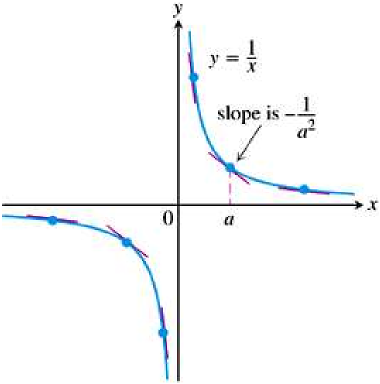
1. Calculate $f(x_0)$ and $f(x_0 + h)$.
2. Calculate the slope
$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$
3. If the limit exists, find the tangent line as
$$y = y_0 + m(x - x_0).$$

Testing the recipe

example: Find slope and tangent to $y = 1/x$ at $x_0 = a \neq 0$

1. $f(a) = \dfrac{1}{a}$, $f(a+h) = \frac{1}{a+h}$
2. slope:

$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{a+h} - \frac{1}{a}}{h} \\ &= \lim_{h\to 0} \frac{a - (a+h)}{h \cdot a(a+h)} \\ &= \lim_{h\to 0} \frac{-1}{a(a+h)} = -\frac{1}{a^2} \end{aligned}$$



2. tangent line at $(a, 1/a)$: $y = 1/a + (-1/a^2)(x - a)$ or

$$y = \frac{2}{a} - \frac{x}{a^2}$$

Derivative

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}$$

.The limit as h approaches 0, if it exists, is called the derivative of f at x_0

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

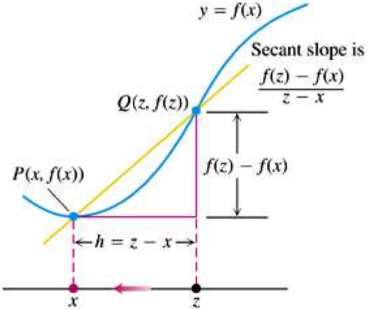
If $f'(x)$ exists, we say that f is differentiable at x.

Equivalent definition and notation

choose $z = x + h$: $h = z - x$ approaches 0 if and only if $z \to x$

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$



equivalent notation: if $y = f(x)$,

$$y' = f'(x) = \frac{d}{dx} f(x) = \frac{dy}{dx}$$

calculating a derivative is called

differentiation

Calculating derivatives from the definition

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h},$$

provided the limit exists.

example: differentiate

$$f(x) = \sqrt{x}$$

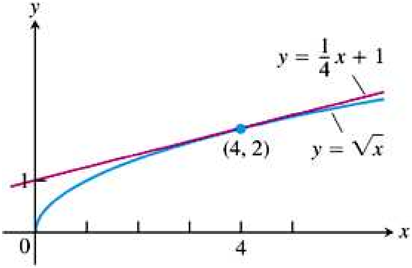
$$f'(x) = \text{[calculation on whiteboard]} = \frac{1}{2\sqrt{x}}$$

Tangent line of the square root function

$$\text{summary: } f(x) = \sqrt{x} \quad \Rightarrow \quad f'(x) = \frac{1}{2\sqrt{x}}$$

calculate the *tangent line* to the curve at $x = 4$:

- $f(4) = 2$, so the line goes through the point $(4, 2)$
- slope $m = f'(4) = 1/4$
- tangent line $y = 2 + m(x - 4)$, i.e.
$$y = \frac{x}{4} + 1$$



note: one sometimes writes

$$f'(4) = \frac{d}{dx}\sqrt{x}\bigg|_{x=4} = \frac{1}{2\sqrt{x}}\bigg|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

Differentiation rules

Rule 1: Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0 .$$

Rule 2: Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1} .$$

Differentiation rules

Rule 3: Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx} .$$

Proof.

$$\begin{aligned}
\frac{d}{dx}cu &= \\
\text{(def. of derivative)} &= \lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h} \\
\text{(limit laws)} &= c\lim_{h\to 0}\frac{u(x+h)-u(x)}{h} \\
(u \text{ is differentiable}) &= c\frac{du}{dx}
\end{aligned}$$

Differentiation rules and their application

Rule 4: Derivative Sum Rule

If u and v are differentiable functions of x, then

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}\ .$$

example: Differentiate $y = x^4 - 2x^2 + 2$.

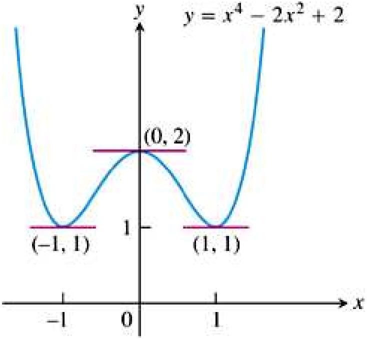
$$\begin{aligned}
\frac{dy}{dx} &= \frac{d}{dx}(x^4 - 2x^2 + 2) \\
\text{(rule 4)} \quad &= \frac{d}{dx}(x^4) + \frac{d}{dx}(-2x^2) + \frac{d}{dx}(2) \\
\text{(rule 3)} \quad &= \frac{d}{dx}(x^4) + (-2)\frac{d}{dx}(x^2) + \frac{d}{dx}(2) \\
\text{(rule 2)} \quad &= 4x^3 + (-2)2x + \frac{d}{dx}(2) \\
\text{(rule 1)} \quad &= 4x^3 - 4x + 0 = 4x^3 - 4x
\end{aligned}$$

Finding horizontal tangents

summary: $y = x^4 - 2x^2 + 2 \quad , \quad y' = 4x^3 - 4x$

Now find, for example, *horizontal tangents*:

$$y' = 4x^3 - 4x = 0 \quad \Rightarrow \quad 4x(x^2 - 1) = 0 \quad \Rightarrow \quad x \in \{0, 1, -1\}$$



Further differentiation rules

Rule 5: Derivative Product Rule

If u and v are differentiable functions of x, then

$$\frac{d}{dx}(uv) = \frac{du}{dx}v + u\frac{dv}{dx} .$$

Rule 6: Derivative Quotient Rule

If u and v are differentiable functions of x and $v(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}v - u\frac{dv}{dx}}{v^2} .$$

Common mistakes:

$$(uv)' = u'v' \quad , \quad (u/v)' = u'/v'$$

is generally **WRONG!**

Using product and quotient rules

examples: (1) Differentiate $y = (x^2+1)(x^3+3)$:

$$\text{use } \boxed{y' = (uv)' = u'v + uv'}$$

$$\text{here: } u = x^2+1\,, \quad v = x^3+3$$

$$u' = 2x\,, \quad v' = 3x^2$$

$$y' = 2x(x^3+3) + (x^2+1)3x^2 = 5x^4 + 3x^2 + 6x$$

(2) Differentiate $y = (t^2-1)/(t^2+1)$:

$$\text{use } \boxed{y' = \left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}}$$

$$\text{here: } u = t^2-1\,, \quad v = t^2+1$$

$$u' = 2t\,, \quad v' = 2t$$

$$y' = \frac{2t(t^2+1)-(t^2-1)2t}{(t^2+1)^2} = \frac{4t}{(t^2+1)^2}$$

Another differentiation rule

Rule 7: Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}x^n = nx^{n-1} \; .$$

[proof: define $n = -m$ and use the quotient rule]

example:

$$\frac{d}{dx}\left(\frac{1}{x^{11}}\right) = \frac{d}{dx}(x^{-11}) = -11x^{-12} \; .$$

Finding higher derivatives

example: Differentiate repeatedly $f(x) = x^5$ and $g(x) = x^{-2}$.

$$
\begin{aligned}
f'(x) &= 5x^4 & g'(x) &= -2x^{-3} \\
f''(x) &= 20x^3 & g''(x) &= 6x^{-4} \\
f'''(x) &= 60x^2 & g'''(x) &= -24x^{-5} \\
f^{(4)}(x) &= 120x & g^{(4)}(x) &= 120x^{-6} \\
f^{(5)}(x) &= 120 & g^{(5)}(x) &= -720x^{-7} \\
f^{(6)}(x) &= 0 & g^{(6)}(x) &= 5040x^{-8} \\
f^{(7)}(x) &= 0 & g^{(7)}(x) &= \dots
\end{aligned}
$$

Derivatives of trigonometric functions

(1) Differentiate $f(x) = \sin x$:

- Start with the **definition** of $f'(x)$:

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

- Use $\sin(x+h) = \sin x \cos h + \cos x \sin h$:

$$f'(x) = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

- Collect terms and apply limit laws:

$$f'(x) = \sin x \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \lim_{h \to 0} \frac{\sin h}{h}$$

- Use $\lim_{h \to 0} \frac{\cos h - 1}{h} = 0$ and $\lim_{h \to 0} \frac{\sin h}{h} = 1$ to conclude

$$f'(x) = \cos x$$

Summary

Derivatives of trigonometric functions

$$
\begin{aligned}
\frac{d}{dx}\sin x &= \cos x \\
\frac{d}{dx}\cos x &= -\sin x \\
\frac{d}{dx}\tan x &= \frac{1}{\cos^2 x} = \sec^2 x \\
\frac{d}{dx}\sec x &= \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \sec x \tan x \\
\frac{d}{dx}\cot x &= \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right) = -\csc^2 x \\
\frac{d}{dx}\csc x &= \frac{d}{dx}\left(\frac{1}{\sin x}\right) = -\csc x \cot x
\end{aligned}
$$

Derivative of composites

example: relating derivatives

$y = \frac{3}{2}x$ is the same as

$$y = \frac{1}{2}u \quad \text{and} \quad u = 3x \; .$$

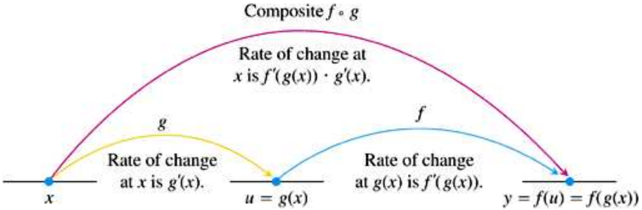
By differentiating

$$\frac{dy}{dx} = \frac{3}{2}\,, \quad \frac{dy}{du} = \frac{1}{2}\,, \quad \frac{du}{dx} = 3$$

we find that

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}\; .$$

The chain rule



THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Applying the chain rule

examples: (1) Differentiate $x(t) = \cos(t^2 + 1)$.

$$\text{use } \boxed{\frac{dx}{dt} = \frac{dx}{du} \cdot \frac{du}{dt}}$$

here: choose $x = \cos u$ and $u = t^2 + 1$ and differentiate,

$$\frac{dx}{du} = -\sin u \quad \text{and} \quad \frac{du}{dt} = 2t \; .$$

Then

$$\frac{dx}{dt} = (-\sin u)2t = \; -2t\sin(t^2 + 1) \; .$$

(2) $\frac{d}{dx} sin(x^2 + x) = \; \cos(x^2 + x)(2x + 1)$

(3) A chain with *three links*:

$$\frac{d}{dt}\tan(5 - \sin 2t) = \; \text{[Details on white board]} \; = \frac{-2\cos 2t}{\cos^2(5 - \sin 2t)} \; .$$

Implicit differentiation

problem: We want to compute y' but do not have an explicit relation $y = f(x)$ available. Rather, we have an implicit relation

$$F(x, y) = 0$$

between x and y.

example:

$$F(x, y) = x^2 + y^2 - 1 = 0 .$$

solutions:

- Use implicit differentiation

Differentiating implicitly

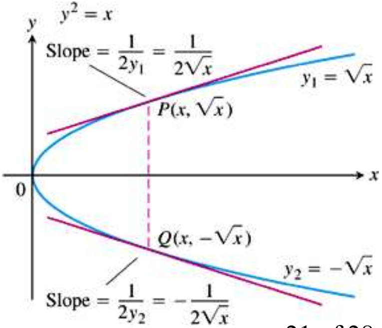
example: Given $y^2 = x$, compute y'.

new method by differentiating *implicitly*:

- Differentiating *both sides* of the equation gives $2yy' = 1$.
- Solving for y' we get $\boxed{y' = \frac{1}{2y}}$.

Compare with differentiating *explicitly*:

- For $y^2 = x$ we have the two *explicit solutions* $|y| = \sqrt{x} \Rightarrow y_{1,2} = \pm\sqrt{x}$ with derivatives $\boxed{y'_{1,2} = \pm\frac{1}{2\sqrt{x}}}$.
- Compare with solution above: substituting $y = y_{1,2} = \pm\sqrt{x}$ therein reproduces the explicit result.



Differentiating implicitly

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

example: the ellipse again, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

1. $\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$
2. $\frac{2yy'}{b^2} = -\frac{2x}{a^2}$
3. $y' = -\frac{b^2}{a^2}\frac{x}{y}$, as obtained via parametrisation in the previous lecture.

Higher-order derivatives

Implicit differentiation also works for higher-order derivatives.

example:

- For the ellipse we had after differentiation:

$$\frac{2x}{a^2} + \frac{2yy'}{b^2} = 0$$

- Differentiate again:

$$\frac{2}{a^2} + \frac{2(y'^2 + yy'')}{b^2} = 0$$

- Now substitute our previous result $y' = -\frac{b^2}{a^2}\frac{x}{y}$ and simplify (this takes a few steps):

$$y'' = -\frac{b^4}{a^2}\frac{1}{y^3} \ ,$$

Extreme values of functions

DEFINITIONS Absolute Maximum, Absolute Minimum

Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

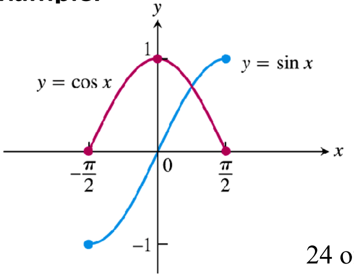
$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

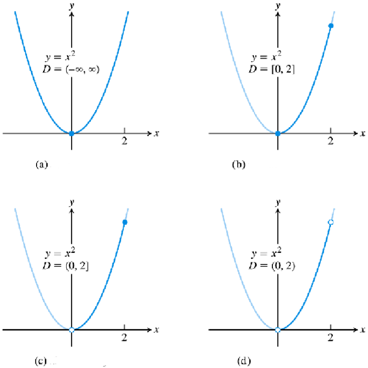
These values are also called absolute **extrema**, or **global** extrema.

example:



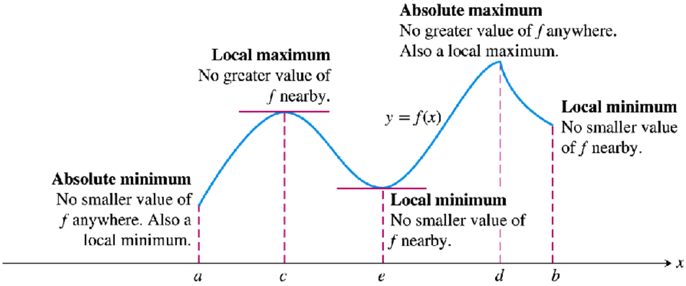
Same rule for different domains yields different extrema

example:



	Domain	abs. max.	abs. min.
(a)	$(-\infty, \infty)$	none	0, at 0
(b)	$[0, 2]$	4, at 2	0, at 0
(c)	$(0, 2]$	4, at 2	none
(d)	$(0, 2)$	none	none

Local (relative) extreme values



DEFINITIONS **Local Maximum, Local Minimum**

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

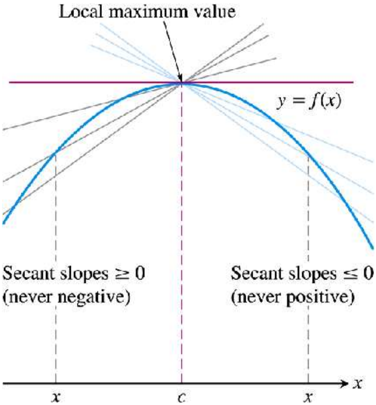
$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

Finding extreme values

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

basic idea of the proof:



Conditions for extreme values

Where can a function f possibly have an extreme value? Recall the

Theorem

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then $f'(c) = 0$.

answer:

1. at interior points where $f' = 0$
2. at interior points where f' is not defined
3. at endpoints of the domain of f.

combine 1 and 2:

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Calculating absolute extrema

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

example 1: Find the absolute extrema of $f(x) = x^2$ on $[-2, 1]$.

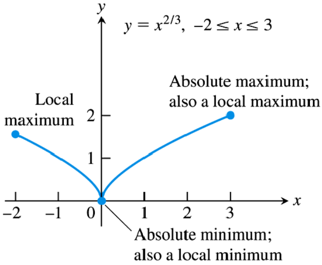
- f is differentiable on $[-2, 1]$ with $f'(x) = 2x$
- critical point: $f'(x) = 0 \quad \Rightarrow \quad x = 0$
- endpoints: $x = -2$ and $x = 1$
- $f(0) = 0$, $f(-2) = 4$, $f(1) = 1$

Therefore f has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$.

Absolute extrema with $f'(c)$ being undefined

example 2: Find the absolute extrema of $f(x) = x^{2/3}$ on $[-2, 3]$.

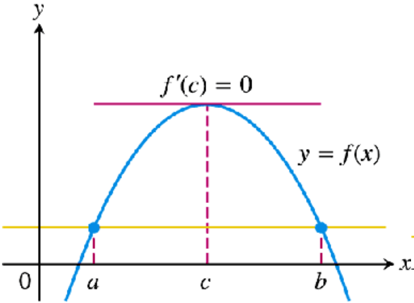
- f is differentiable with $f'(x) = \frac{2}{3}x^{-1/3}$ *except* at $x = 0$
- critical point: $f'(x) = 0$ or $f'(x)$ undefined $\Rightarrow \quad x = 0$
- endpoints: $x = -2$ and $x = 3$
- $f(-2) = \sqrt[3]{4}$, $f(0) = 0$, $f(3) = \sqrt[3]{9}$

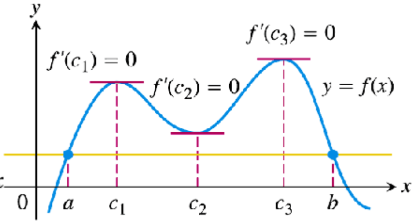


Therefore f has an absolute maximum value of $\sqrt[3]{9}$ at $x = 3$ and an absolute minimum value of 0 at $x = 0$.

Rolle's theorem

motivation:





Theorem

Let $f(x)$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$ then there exists a $c \in (a, b)$ with

$$f'(c) = 0 \,.$$

Assumptions in Rolle's theorem

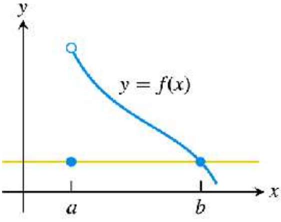
Theorem

Let $f(x)$ be continuous on $[a,b]$ and differentiable on (a,b). If $f(a)=f(b)$ then there exists a $c \in (a,b)$ with

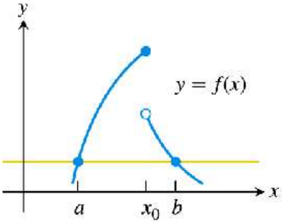
$$f'(c) = 0\ .$$

It is essential that all of the hypotheses in the theorem are fulfilled!

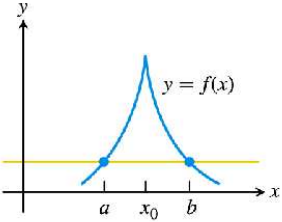
examples:



(a) Discontinuous at an endpoint of $[a, b]$



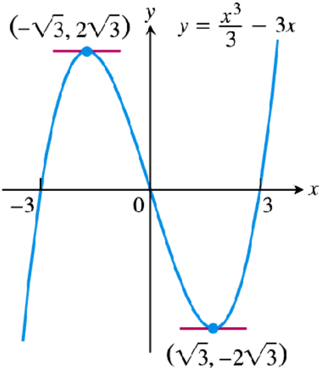
(b) Discontinuous at an interior point of $[a, b]$



(c) Continuous on $[a, b]$ but not differentiable at an interior point

Horizontal tangents of a cubic polynomial

example: Apply Rolle's theorem to $f(x) = \frac{x^3}{3} - 3x$ on $[-3, 3]$.



- polynomial f is continuous on $[-3, 3]$ and differentiable on $(-3, 3)$
- $f(-3) = f(3) = 0$
- by Rolle's theorem there exists (at least!) one $c \in [-3, 3]$ with $f'(c) = 0$

From $f'(x) = x^2 - 3 = 0$ we find that indeed $x = \pm\sqrt{3}$.

Increasing and decreasing functions

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).
If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Increasing and decreasing functions

example: Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

$$f'(x) = 3x^2 - 12 = 3(x^2 - 4) = 3(x+2)(x-2)$$

$$\Rightarrow x_1 = -2, x_2 = 2$$

These critical points subdivide the natural domain into

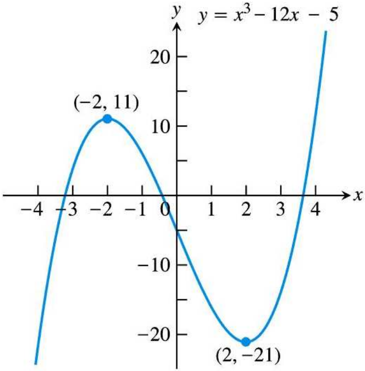
$$(-\infty, -2), (-2, 2), (2, \infty) \ .$$

rule: If $a < b$ are two nearby critical points for f, then f' must be positive on (a, b) *or* negative there. (proof relies on continuity of f').This implies that for finding the sign of f' it suffices to compute $f'(x)$ at *one* $x \in (a, b)$!

$$\text{here: } f'(-3) = 15 \ , \ f'(0) = -12 \ , \ f'(3) = 15$$

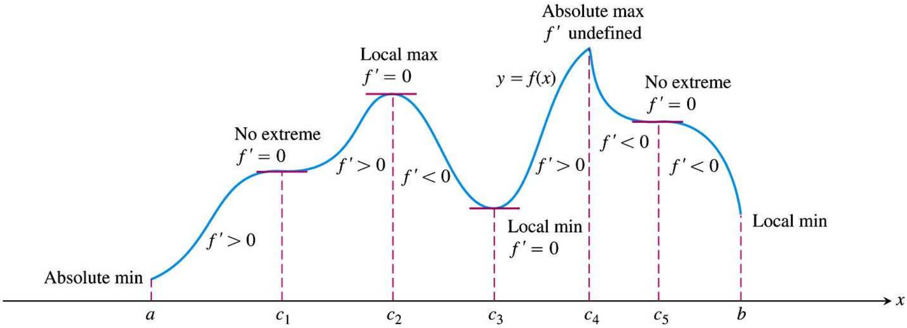
Increasing and decreasing functions

intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
sign of f'	$+$	-	$+$
behaviour of f	increasing	decreasing	increasing



First derivatives and local extrema

example:



- whenever f has a minimum, $f' < 0$ to the left and $f' > 0$ to the right
- whenever f has a maximum, $f' > 0$ to the left and $f' < 0$ to the right

$\Rightarrow$ At local extrema, the sign of $f'(x)$ changes!

First derivatives and local extrema

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

Using the first derivative test for local extrema

example: Find the critical points of $f(x) = x^{4/3} - 4x^{1/3}$. Identify the intervals on which f is increasing and decreasing. Find the function's extrema.

$$f'(x) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3} = \frac{4}{3}\frac{x-1}{x^{2/3}}$$

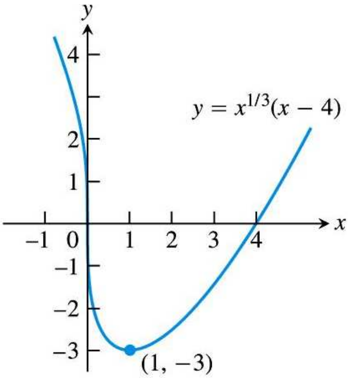
$$\Rightarrow x_1 = 1\ ,\ x_2 = 0$$

intervals	$x < 0$	$0 < x < 1$	$1 < x$
sign of f'	-	-	$+$
behaviour of f	decreasing	decreasing	increasing

Apply the first derivative test to identify local extrema:

- f' does not change sign at $x = 0 \Rightarrow$ no extremum
- f' changes from $-$ to $+$ $\Rightarrow$ local minimum

Summary: geometrical picture

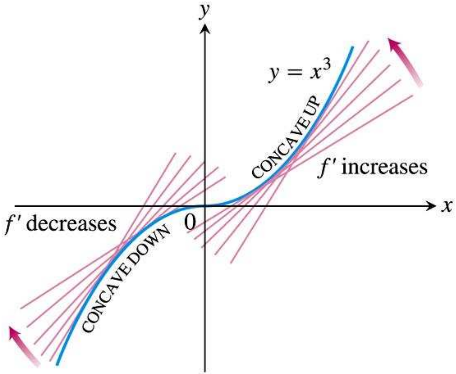


Since $\lim_{x\to\pm\infty} = \infty$, the minimum at $x = 1$ with $f(1) = -3$ is also an *.absolute minimum*

Note that $f'(0) = -\infty$!

Concavity of a function

example:



intervals	$x < 0$	$0 < x$
turning of curve	turns to the *right*	turns to the *left*
tangent slopes	decreasing	increasing

The turning or bending behaviour defines the concavity of the curve

Testing for concavity

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

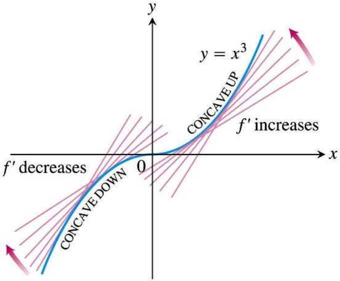
1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

Applying the concavity test

example 1:

$$y = x^3 \Rightarrow y'' = 6x$$

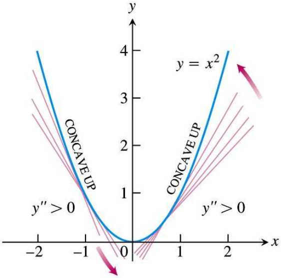
for $(-\infty, 0)$ it is $y'' < 0$:graph concave down;
for $(0, \infty)$ it is $y'' < 0$:
graph concave up



example 2:

$$y = x^2 \Rightarrow y'' = 2 > 0$$

graph is concave up everywhere



Second derivatives at extrema

Look at second derivative instead of sign changes at critical points in order to test for local extrema:

THEOREM 5 Second Derivative Test for Local Extrema

Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Summary: curve sketching

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

Application: curve sketching

example: Sketch the graph of $f(x) = \frac{(x+1)^2}{1+x^2}$.

1. The natural domain of f is$(-\infty, \infty)$; no symmetries about any axis.
2. calculate derivatives:

$$f'(x) = \text{[calculation on whiteboard]} = \frac{2(1-x^2)}{(1+x^2)^2}$$

$$f''(x) = \text{[calculation on whiteboard]} = \frac{4x(x^2-3)}{(1+x^2)^3}$$

3. critical points: f' exists on $(-\infty, \infty)$ with $f'(\pm 1) = 0$ and $f''(-1) = 1 > 0$, $f''(1) = -1 < 0$:
   $(-1, 0)$ is a local minimum and $(1, 2)$ a local maximum.

Example continued 1

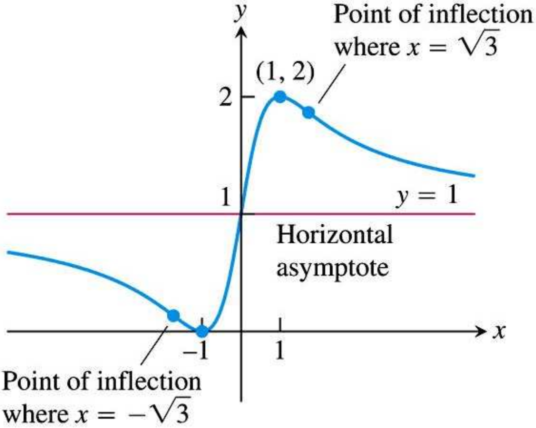
4. On $(-\infty,-1)$ it is $f'(x)<0$: curve decreasing; on $(-1,1)$ it is $f'(x)>0$: curve increasing; on $(1,\infty)$ it is $f'(x)<0$: curve decreasing
5. $f''(x)=0$ if $x=\pm\sqrt{3}$ or 0; $f''<0$ on $(-\infty,-\sqrt{3})$: concave down; $f''>0$ on $(-\sqrt{3},0)$: concave up; $f''<0$ on $(0,\sqrt{3})$: concave down; $f''>0$ on $(\sqrt{3},\infty)$: concave up. Each point is a point of inflection.
6. calculate asymptotes:

$$f(x)=\frac{(x+1)^2}{1+x^2}=\frac{x^2+2x+1}{1+x^2}=\frac{1+2/x+1/x^2}{1/x^2+1}$$

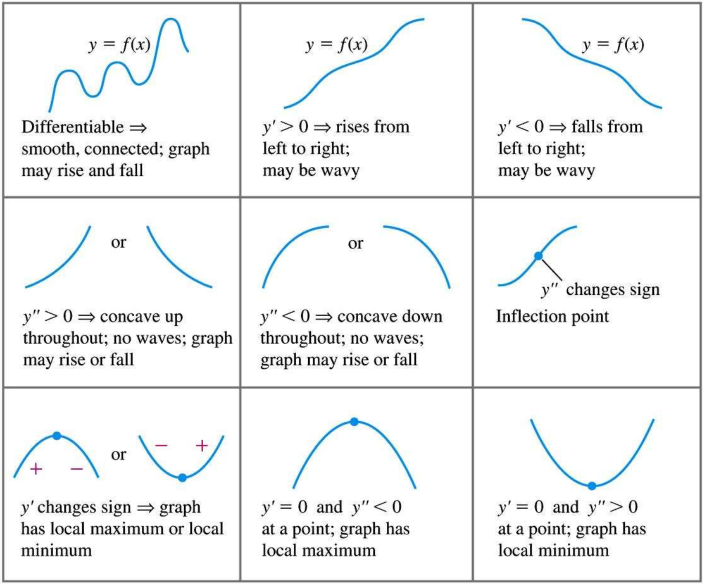
$f(x)\to 1^+$ as $x\to\infty$ and $f(x)\to 1^-$ as $x\to-\infty$: $y=1$ is a horizontal asymptote. No vertical asymptotes.

Example continued 2

8. sketch the curve:



Summary


$y = f(x)$
Differentiable ⇒ smooth, connected; graph may rise and fall
$y = f(x)$
$y' > 0$ ⇒ rises from left to right; may be wavy
$y = f(x)$
$y' < 0$ ⇒ falls from left to right; may be wavy
or
$y'' > 0$ ⇒ concave up throughout; no waves; graph may rise or fall
or
$y'' < 0$ ⇒ concave down throughout; no waves; graph may rise or fall
y'' changes sign
Inflection point
+ −
or
− +
y' changes sign ⇒ graph has local maximum or local minimum
$y' = 0$ and $y'' < 0$ at a point; graph has local maximum
$y' = 0$ and $y'' > 0$ at a point; graph has local minimum

Mathematics (MATH113)

Week 5 (Lectures No. 9 & 10)

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2024

L'Hôpital's Rule

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces 0/0, a meaningless expression, which we cannot evaluate.

L'Hôpital's Rule

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Caution

To apply l'Hôpital's Rule to f/g, the derivative of f divide by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$

L'Hôpital's Rule

EXAMPLE 1 Using L'Hôpital's Rule

(a) $$\lim_{x\to 0} \frac{3x - \sin x}{x} = \left.\frac{3 - \cos x}{1}\right|_{x=0} = 2$$

(b) $$\lim_{x\to 0} \frac{\sqrt{1 + x} - 1}{x} = \left.\frac{\frac{1}{2\sqrt{1+x}}}{1}\right|_{x=0} = \frac{1}{2}$$

Sometimes after differentiation, the new numerator and denominator both equal zero at $x = a$, as we see in Example 2. In these cases, we apply a stronger form of l'Hôpital's Rule

L'Hôpital's Rule

THEOREM 7 L'Hôpital's Rule (Stronger Form)

Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

L'Hôpital's Rule

EXAMPLE 2 Applying the Stronger Form of L'Hôpital's Rule

(a) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\qquad \frac{0}{0}$

$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$ $\qquad$ Still $\frac{0}{0}$; differentiate again.

$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$ $\qquad$ Not $\frac{0}{0}$; limit is found.

(b) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$ $\qquad \frac{0}{0}$

$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$ $\qquad$ Still $\frac{0}{0}$

$= \lim_{x\to 0} \frac{\sin x}{6x}$ $\qquad$ Still $\frac{0}{0}$

$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$ $\qquad$ Not $\frac{0}{0}$; limit is found.

L'Hôpital's Rule

EXAMPLE 3 Incorrectly Applying the Stronger Form of L'Hôpital's Rule

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x\to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x\to 0} \frac{\sin x}{1 + 2x} = \lim_{x\to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is wrong. L'Hôpital's Rule can only be applied to limits which give indeterminate forms, and $0/1$ is not an indeterminate form

L'Hôpital's Rule

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x}$$

Common denominator is $x \sin x$

Then apply l'Hôpital's Rule to the result:

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) = \lim_{x \to 0} \frac{x - \sin x}{x \sin x} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} \qquad \text{Still } \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0.$$

Natural Logarithm

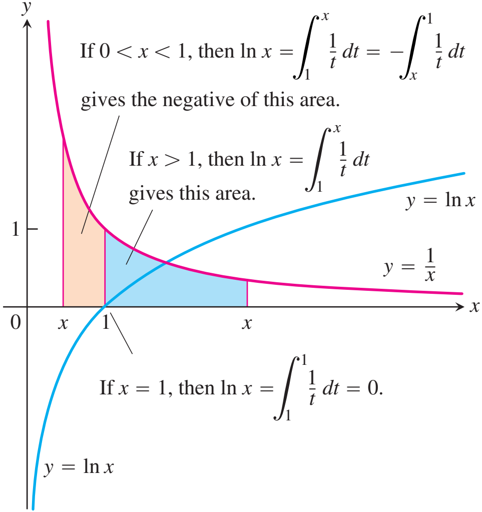
DEFINITION The Natural Logarithm Function

$$\ln x = \int_1^x \frac{1}{t}\, dt, \qquad x > 0$$

If $x > 1$, then $\ln x$ is the area under the curve $y = 1/t$ from $t = 1$ to $t = x$ (Figure 7.9). For $0 < x < 1$, $\ln x$ gives the negative of the area under the curve from x to 1. The function is not defined for $x \leq 0$. From the Zero Width Interval Rule for definite integrals, we also have

$$\ln 1 = \int_1^1 \frac{1}{t}\, dt = 0.$$

Natural Logarithm



Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ — Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ — r rational

Properties of Logarithms

$$\ln 0 = -\infty$$

$$\ln 1 = 0$$

$$\ln e = 1$$

The Derivative of $y = \ln x$

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0$$

Using Logarithmic Differentiation

Find $dy\ \ dx$ if

$$y = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}, \qquad x > 1.$$

Solution We take the natural logarithm of both sides and simplify the result with the properties of logarithms:

$$\begin{aligned}
\ln y &= \ln \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1} \\
&= \ln\left((x^2 + 1)(x + 3)^{1/2}\right) - \ln(x - 1) && \text{Rule 2} \\
&= \ln(x^2 + 1) + \ln(x + 3)^{1/2} - \ln(x - 1) && \text{Rule 1} \\
&= \ln(x^2 + 1) + \frac{1}{2}\ln(x + 3) - \ln(x - 1). && \text{Rule 3}
\end{aligned}$$

Using Logarithmic Differentiation

We then take derivatives of both sides with respect to x, using Equation (1) on the left:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2 + 1} \cdot 2x + \frac{1}{2} \cdot \frac{1}{x + 3} - \frac{1}{x - 1}.$$

Next we solve for dy/dx:

$$\frac{dy}{dx} = y\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

Finally, we substitute for y:

$$\frac{dy}{dx} = \frac{(x^2 + 1)(x + 3)^{1/2}}{x - 1}\left(\frac{2x}{x^2 + 1} + \frac{1}{2x + 6} - \frac{1}{x - 1}\right).$$

The Exponential Function

$$\ln e^r = r \ln e = r \cdot 1 = r$$

DEFINITION The Natural Exponential Function

For every real number x, $e^x = \ln^{-1} x = \exp x$.

Inverse Equations for e^x and $\ln x$

$$e^{\ln x} = x \qquad (\text{all } x > 0)$$

$$\ln(e^x) = x \qquad (\text{all } x)$$

The Exponential Function

DEFINITION General Exponential Functions

For any numbers $a > 0$ and x, the exponential function with base a is

$$a^x = e^{x \ln a}.$$

When $a = e$, the definition gives $a^x = e^{x \ln a} = e^{x \ln e} = e^{x \cdot 1} = e^x$.

Laws of Exponents for e^x

For all numbers x, x_1, and x_2, the natural exponential e^x obeys the following laws:

1. $e^{x_1} \cdot e^{x_2} = e^{x_1+x_2}$

2. $e^{-x} = \frac{1}{e^x}$

3. $\frac{e^{x_1}}{e^{x_2}} = e^{x_1-x_2}$

4. $(e^{x_1})^{x_2} = e^{x_1x_2} = (e^{x_2})^{x_1}$

The Derivative of e^x

$$\frac{d}{dx} e^x = e^x$$

$$\frac{d}{dx}(5e^x) = 5\frac{d}{dx} e^x$$

$$= 5e^x$$

If u is any differentiable function of x, then

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

Applying the Chain Rule with Exponentials

(a) $\frac{d}{dx} e^{-x} = e^{-x} \frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}$ with $u = -x$

(b) $\frac{d}{dx} e^{\sin x} = e^{\sin x} \frac{d}{dx}(\sin x) = e^{\sin x} \cdot \cos x$ with $u = \sin x$

a^u

The Derivative of a^u

We start with the definition $a^x = e^{x \ln a}$:

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a) \qquad \frac{d}{dx} e^u = e^u \frac{du}{dx}$$

$$= a^x \ln a .$$

If $a > 0$, th

$$\frac{d}{dx} a^x = a^x \ln a .$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx} a^u = a^u \ln a \, \frac{du}{dx} .$$

Differentiating a General Power Function

Find dy/dx if $y = x^x, \quad x > 0.$

Solution Write x^x as a power of e:

$$y = x^x = e^{x \ln x}. \qquad a^x \text{ with } a = x.$$

Then differentiate as usual:

$$\frac{dy}{dx} = \frac{d}{dx} e^{x \ln x}$$

$$= e^{x \ln x} \frac{d}{dx} (x \ln x)$$

$$= x^x \left(x \cdot \frac{1}{x} + \ln x \right)$$

$$= x^x (1 + \ln x).$$

Logarithms with Base a ($\log_a x$)

DEFINITION $\log_a x$

For any positive number $a \neq 1$,

$$\log_a x \text{ is the inverse function of } a^x.$$

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$

$$\log_a x = \frac{1}{\ln a} \cdot \ln x = \frac{\ln x}{\ln a}$$

The Derivative of $(\log_a x)$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms

If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}}$$

Mathematics (MATH113)

Week 6 (Lectures No. 11 & 12)

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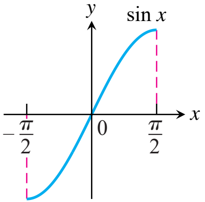
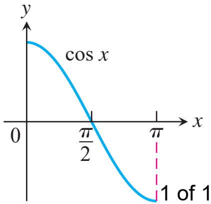
2024

Inverse Trigonometric Functions

Domain restrictions that make the trigonometric functions one-to-one

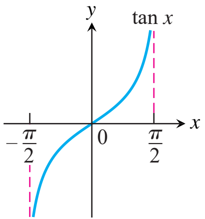
Function	**Domain**	**Range**
sin x	$[-\pi/2, \pi/2]$	$[-1, 1]$
cos x	$[0, \pi]$	$[-1, 1]$

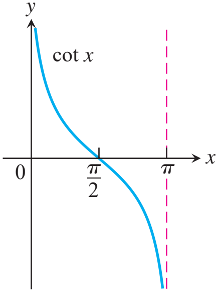




Inverse Trigonometric Functions

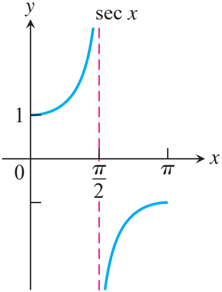
Function	Domain	Range
tan x	$(-\pi/2, \pi/2)$	$(-\infty, \infty)$
cot x	$(0, \pi)$	$(-\infty, \infty)$

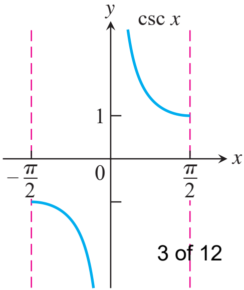




Inverse Trigonometric Functions

Function	Domain	Range
sec x	$[0, \pi/2) \cup (\pi/2, \pi]$	$(-\infty, -1] \cup [1, \infty)$
csc x	$[-\pi/2, 0) \cup (0, \pi/2]$	$(-\infty, -1] \cup [1, \infty)$





Inverse Trigonometric Functions

Since these restricted functions are now one-to-one, they have inverses, which we denote by

$$y = \sin^{-1} x \quad \text{or} \quad y = \arcsin x$$
$$y = \cos^{-1} x \quad \text{or} \quad y = \arccos x$$
$$y = \tan^{-1} x \quad \text{or} \quad y = \arctan x$$
$$y = \cot^{-1} x \quad \text{or} \quad y = \operatorname{arccot} x$$
$$y = \sec^{-1} x \quad \text{or} \quad y = \operatorname{arcsec} x$$
$$y = \csc^{-1} x \quad \text{or} \quad y = \operatorname{arccsc} x$$

These equations are read "y equals the arcsine of x" or "y equals arcsin x" and so on

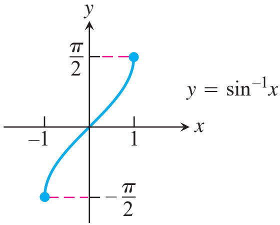
Inverse Trigonometric Functions

CAUTION The -1 “.in the expressions for the inverse means “inverse
It does *not* mean reciprocal

For example, the *reciprocal* of $\sin x$ is $(\sin x)^{-1} = 1/\sin x = \csc x$.

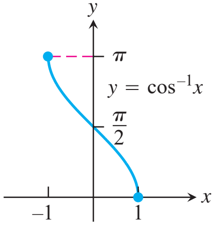
Inverse Trigonometric Functions

Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



(a)

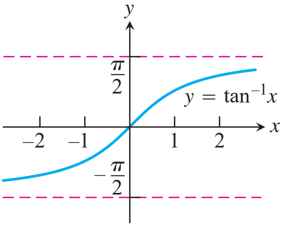
Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



(b)

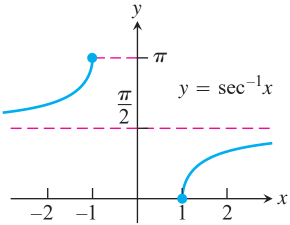
Inverse Trigonometric Functions

Domain: $-\infty < x < \infty$
Range: $-\dfrac{\pi}{2} < y < \dfrac{\pi}{2}$



(c)

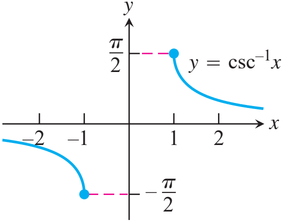
Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \dfrac{\pi}{2}$



(d)

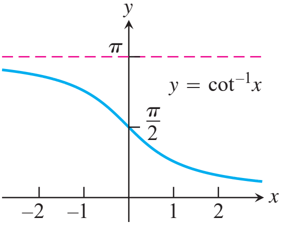
Inverse Trigonometric Functions

Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$



(e)

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



(f)

Inverse Trigonometric Functions

$$\sin^{-1}(-x) = -\sin^{-1} x$$

The graph of $y = \cos^{-1} x$ has no such symmetry

$$\cos^{-1} x + \cos^{-1}(-x) = \pi,$$

$$\cos^{-1}(-x) = \pi - \cos^{-1} x.$$

$$\sin^{-1} x + \cos^{-1} x = \pi/2.$$

Inverse Trigonometric Functions

x	$\sin^{-1} x$
$\sqrt{3}/2$	$\pi/3$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/6$
$-1/2$	$-\pi/6$
$-\sqrt{2}/2$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

x	$\cos^{-1} x$
$\sqrt{3}/2$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$
$1/2$	$\pi/3$
$-1/2$	$2\pi/3$
$-\sqrt{2}/2$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$

x	$\tan^{-1} x$
$\sqrt{3}$	$\pi/3$
1	$\pi/4$
$\sqrt{3}/3$	$\pi/6$
$-\sqrt{3}/3$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Inverse Trigonometric Functions

$$\sin y = x \qquad y = \sin^{-1} x \Leftrightarrow \sin y = x$$

$$\frac{d}{dx}(\sin y) = 1 \qquad \text{Derivative of both sides with respect to } x$$

$$\cos y \, \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$

$$\frac{dy}{dx} = \frac{1}{\cos y} \qquad \text{We can divide because } \cos y > 0 \text{ for } -\pi/2 < y < \pi/2.$$

$$= \frac{1}{\sqrt{1 - x^2}} \qquad \cos y = \sqrt{1 - \sin^2 y}$$

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1 - x^2}}.$$

Derivatives of the inverse trigonometric functions

1. $$\frac{d(\sin^{-1} u)}{dx} = \frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

2. $$\frac{d(\cos^{-1} u)}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$$

3. $$\frac{d(\tan^{-1} u)}{dx} = \frac{du/dx}{1+u^2}$$

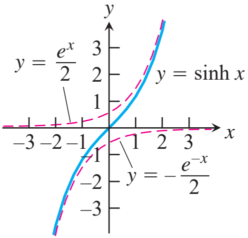
4. $$\frac{d(\cot^{-1} u)}{dx} = -\frac{du/dx}{1+u^2}$$

5. $$\frac{d(\sec^{-1} u)}{dx} = \frac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

6. $$\frac{d(\csc^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$$

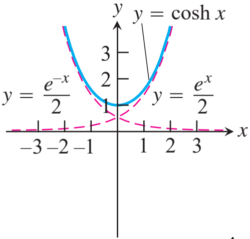
The Six Basic Hyperbolic Functions

Hyperbolic sine of x: $$\sinh x = \frac{e^x - e^{-x}}{2}$$



(a)

Hyperbolic cosine of x: $$\cosh x = \frac{e^x + e^{-x}}{2}$$

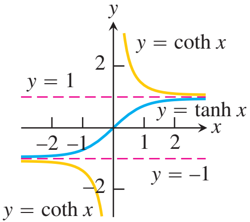


(b)

The Six Basic Hyperbolic Functions

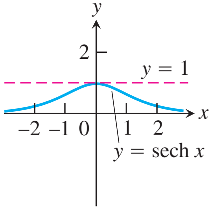
Hyperbolic tangent: $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

Hyperbolic cotangent: $\coth x = \dfrac{\cosh x}{\sinh x} = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$

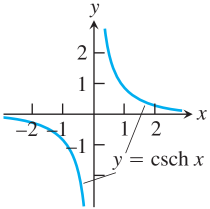


The Six Basic Hyperbolic Functions

Hyperbolic secant: $$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



Hyperbolic cosecant: $$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$



Properties of Hyperbolic Functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

Properties of Hyperbolic Functions

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivative of Hyperbolic Functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Derivative of Hyperbolic Functions

The derivative formulas are derived from the derivative of e^u:

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) \qquad \text{Definition of } \sinh u$$

$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} \qquad \text{Derivative of } e^u$$

$$= \cosh u \frac{du}{dx} \qquad \text{Definition of } \cosh u$$

Derivative of Hyperbolic Functions

This gives the first derivative formula. The calculation

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of csch } u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of csch } u \text{ and coth } u
\end{aligned}$$

gives the last formula. The others are obtained similarly.

Derviative of Hyperbolic Functions

$$\frac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1=t^2}\cdot\frac{d}{dt}\left(\sqrt{1+t^2}\right)$$

$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

Inverse Hyperbolic Functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1 + u^2}} \frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2 - 1}} \frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1 - u^2} \frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1 - u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1 + u^2}}, \qquad u \neq 0$$